Well-foundedness, Induction, Slim Neumann and Frege Ordinals and Least and Greatest Fixpoints of Monotonic Functors: Knaster–Tarski, Aczel, &c.

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Abstract

Firstly, we show that the Neumann ordinals can be defined and understood in set theory without fixing a special theory of sets and classes such as Neumann–Bernays–Gödel, Morse–Kelley, or Quine's ML, and without any axioms, but the Axiom of Extensionality. Especially, no axioms of choice, foundation, infinity, subset, or power are required.
Secondly, for general monotonic functors we present Knaster–Tarski and *well-ordered fixpoint construction*. For *set-continuous monotonic class operators* we present least and greatest fixpoint construction in set theory. For *algebraic class operators* there is a special of construction of the elements of the least fixpoint with labeled well-founded rooted graphs. As special monotonic class operators we discuss *closure operators* and their relation to *complete lattices*, as well as *algebraic closure operators* and their relation to *algebraic lattices*. Finally, we show how to construct two monotonic closure operators from a monotonic functor, namely the *least-fixpoint* and the *greatest-fixpoint* operator.

Contents

1 Urelements, Sets, and Classes

1.1 The Elementary Signature

The objects of class theory are partitioned into *classes* and *urelements*. The classes are again partitioned into *sets* and *proper classes*.

These partitions develop out of the binary element predicate "$\_ \in \_$" and the singulary predicate symbol of urelementship "$\mathcal{U}(\_)$", which make up the whole *elementary* (i.e. non-defined) signature of class theory, besides the pair constructor "$(\_,\_)$", which may be reduced to classes or urelements in some class theories, but which should always be included for conceptual reasons anyway.

Following [Tarski, 1986], we do not consider the binary predicate of *primitive equality* "$\_=\_$" as a part of any signature, but of our logic language already.

1.2 Class Comprehension

It is convenient to add a class constructor $\{\ x \mid A\ \}$, which binds a variable x for a formula A and, roughly speaking, is intended to construct "the class of all x such that A" as a result of this description and comprehension.

In Requirement 1.1, we assume the axiom scheme of *class* comprehension which is common to the class theories of MORSE–KELLEY (MK) and QUINE's ML (ML).[1] It differs from the one of NEUMANN–BERNAYS–GÖDEL (NBG) only insofar as it admits binding of unrestricted class variables in the formula A. As there are no proper classes but only sets in the set theories of ZERMELO–FRAENKEL (ZF) (similar to NBG and MK) and QUINE's NF (NF) (similar to ML), ZF and NF do not have any class comprehension schemes.

Requirement 1.1 (Axiom of Class Comprehension)
For each formula A, each variable x, and each term t, we require: then we require:

$$t \ \in\ \{\ x \mid A\ \} \qquad \text{iff} \qquad A\{x{\mapsto}t\}\ \wedge\ \exists Z.\ (t \in Z).$$

In § 1.6, we will show that we can eliminate the class constructor from all formulas, so that we do not have to add it to the symbols of our elementary signature.

We can now already use the class constructor to define the following:

$$\begin{array}{rcll}
\emptyset & := & \{\ y \mid y \neq y\ \} & \text{"empty set"} \\
\mathcal{V} & := & \{\ y \mid y = y\ \} & \text{"universal class"} \\
\mathcal{S} & := & \{\ y \mid \neg\mathcal{U}(y)\ \} & \text{"class of all sets"} \\
\{X\} & := & \{\ y \mid y = X\ \} & \text{"singleton set of } X\text{"} \\
\mathcal{R} & := & \{\ x \mid x \notin x\ \} & \text{"R{\scriptsize USSELL} class"} \\
X \cap Y & := & \{\ x \mid x \in X \wedge x \in Y\ \} & \text{"union of } X \text{ and } Y\text{"} \\
X \cup Y & := & \{\ x \mid x \in X \vee x \in Y\ \} & \text{"intersection of } X \text{ and } Y\text{"} \\
X \setminus Y & := & \{\ x \mid x \in X \wedge x \notin Y\ \} & \text{"complement of } Y \text{ w.r.t. } X\text{"} \\
\mathrm{dom}(R) & := & \{\ x \mid \exists y.\ ((x,y) \in R)\ \} & \text{"domain of } R\text{"} \\
\mathrm{ran}(R) & := & \{\ y \mid \exists x.\ ((x,y) \in R)\ \} & \text{"range of } R\text{"} \\
\mathrm{field}(R) & := & \mathrm{dom}(R) \cup \mathrm{ran}(R) & \text{"field of } R\text{"}
\end{array}$$

Corollary 1.2 $\exists Z.\ (t \in Z)$ *iff* $t \in \mathcal{V}$.

Corollary 1.3 *For each formula A, each variable x, and each term t, we have:*

$$t \ \in \ \{\ x \mid A\ \} \quad \text{iff} \quad A\{x{\mapsto}t\} \ \wedge \ t \in \mathcal{V}.$$

For t being not an object variable but a term with precisely the new variables $x_1, \ldots, x_n$, the notation "$\{\ t \mid A\ \}$" abbreviates $\{\ y \mid \exists x_1, \ldots, x_n.\ (y{=}t \wedge A)\ \}$, for a new variable y. Similarly, "$\{\ t \in B \mid A\ \}$" abbreviates $\{\ y \mid \exists x_1, \ldots, x_n.\ (y{=}t \wedge y \in B \wedge A)\ \}$. We now can define the following:

$$\begin{array}{rcll}
\mathrm{id} & := & \{\ (x,x) \mid x = x\ \} & \text{"identity function"} \\
X{\times}Y & := & \{\ (x,y) \mid x \in X \wedge x \in Y\ \} & \text{"Cartesian product of } X \text{ and } Y\text{"} \\
R^{-1} & := & \{\ (y,x) \mid (x,y) \in R\ \} & \text{"reverse relation of } R\text{"} \\
R \circ S & := & \{\ (x,z) \mid \exists y.\ ((x,y) \in R \wedge (y,z) \in S)\ \} & \text{"concatenation of } R \text{ and } S\text{"} \\
{}_X{\upharpoonleft}R & := & \{\ (x,y) \in R \mid x \in X\ \} & \text{"restriction of } R \text{ to } X\text{"} \\
R{\upharpoonright}_Y & := & \{\ (x,y) \in R \mid y \in Y\ \} & \text{"range-restriction of } R \text{ to } Y\text{"} \\
\langle X \rangle R & := & \{\ y \mid \exists x \in X.\ ((x,y) \in R)\ \} & \text{"image of } X \text{ under } R\text{"} \\
 & = & \mathrm{ran}({}_X{\upharpoonleft}R) & \\
R\langle Y \rangle & := & \{\ x \mid \exists y \in Y.\ (x,y) \in R\ \} & \text{"reverse-image of } Y \text{ under } R\text{"} \\
 & = & \mathrm{dom}(R{\upharpoonright}_Y) & \\
R(x) & := & \varepsilon y.\ ((x,y) \in R) & \text{"functional application"}
\end{array}$$

The precise definition of functional application is not really important here because we will write "$R(x)$" only if $\exists z.\ \forall y.\ (y = z \ \Leftrightarrow \ (x,y) \in R)$. The reason for a concrete definition is that we need it for explaining stratifiedness, cf. Definition 1.30. The reason for choosing a descriptive term is that we want to avoid overspecification. And the reason for choosing HILBERT's ε-operator instead of PEANO's (inverted) ι-operator is because we think ι-operators to be obsolete, cf. [Wirth, 2008]. Only to be self-contained here, we assume the following axiom scheme for the ε:

Requirement 1.4 (ε-Formula)
For each formula A, each variable x, and each term t, we require:

$$A\{x{\mapsto}t\} \quad \Rightarrow \quad A\{x \mapsto \varepsilon x.\ A\}.$$

1.3 Urelements

For very good reasons,[2] we have chosen urelements to be an elementary concept in our class theory. And we denote the singulary predicate of being an urelement by the symbol "$\mathcal{U}(\_)$" of our elementary signature of class theory; cf. § 1.1.

In the literature, urelements are sometimes called "atoms". This, however, is misleading because urelements do not have to be atomic. For example, a pair of two sets may well be an urelement.

To the contrary, the German prefix "ur" indicates exactly[3] the proper intention behind urelements: *Ur*elements are *neither constructed* by the set constructor $\{\ x \mid A\ \}$ *nor derived* by comprehension (cf. Requirement 1.5). Instead, *urelements* are possibly given as additional *elements* (i.e. no proper classes) of *unknown origin*.

As the most basic intention related to urelements is to be different from classes, we capture this intention with the following axiom scheme:

Requirement 1.5 (Axiom of Urelements)
For each formula A and each variable x, we require $\neg\mathcal{U}(\{\ x \mid A\ \})$.

For discussion of equality in § 1.4, we also need the following relation symbol:

Definition 1.6 (Equality of Urelements "$\_ =_{\mathcal{U}} \_$")
$X =_{\mathcal{U}} Y \quad$ if $\quad X = Y \ \wedge\ \mathcal{U}(X)\ \wedge\ \mathcal{U}(Y)$.

1.4 Subclass and Extensionality

Definition 1.7 (Subclass, '$\subseteq$', '$\supseteq$')
$X \subseteq Y$ if $\neg\mathcal{U}(X) \ \wedge \ \neg\mathcal{U}(Y) \ \wedge \ \forall x.\ (x \in X \Rightarrow x \in Y)$.
$Y \supseteq X$ if $X \subseteq Y$.
$X \subsetneq Y$ if $X \subseteq Y \ \wedge \ X \neq Y$.

As corollaries of Requirement 1.5 and Definition 1.7 we get:

Corollary 1.8 $\emptyset \subseteq \mathcal{S} \subseteq \mathcal{V}$.

Corollary 1.9
The following three statements are logically equivalent: $\neg\mathcal{U}(X)$; $\emptyset \subseteq X$; $X \subseteq \mathcal{V}$.

Definition 1.7 makes $\subseteq$ reflexive on non-urelements (i.e. classes) and transitive. We will assume antisymmetry of $\subseteq$ as an axiom, cf. Requirement 1.10. Thus, the subclass relation $\subseteq$ will be a reflexive ordering on classes.

Requirement 1.10 (Axiom of Extensionality)
$\forall X, Y.\ (X \subseteq Y \wedge X \supseteq Y \ \Rightarrow \ X = Y)$.

Lemma 1.11

$$X = Y \quad \Leftrightarrow \quad \textbf{if} \left(\begin{array}{rl} & \mathcal{U}(X) \\ \vee & \mathcal{U}(Y) \end{array} \right) \textbf{ then } X =_{\mathcal{U}} Y \textbf{ else } \forall x. \left(\begin{array}{rl} & x \in X \\ \Leftrightarrow & x \in Y \end{array} \right) \textbf{ fi}.$$

Remark 1.12 If we assume "$\_ =_{\mathcal{U}} \_$" to be an elementary symbol of our language, then we can read Lemma 1.11 as a definition and use it to remove all occurrences of "$\_=\_$", which then can be treated as a defined symbol.

Proof of Lemma 1.11

$\underline{\mathcal{U}(X) \wedge \mathcal{U}(Y)}$: In this case, $X = Y$ is logically equivalent to $X =_{\mathcal{U}} Y$ according to Definition 1.6.

$\underline{\mathcal{U}(X) \wedge \neg\mathcal{U}(Y)}$: In this case, $X =_{\mathcal{U}} Y$ is logically equivalent to false according to Definition 1.6. Moreover, in this case, $X = Y$ implies $\mathcal{U}(X) \wedge \neg\mathcal{U}(X)$, which implies false. The other direction is trivial: *Ex falso quodlibet!*

$\underline{\neg\mathcal{U}(X) \wedge \mathcal{U}(Y)}$: This case is symmetric to the previous one.

$\underline{\neg\mathcal{U}(X) \wedge \neg\mathcal{U}(Y)}$: In this case, $X = Y$ trivially implies $\forall x.\ (x \in X \Leftrightarrow x \in Y)$. On the other hand, in the latter case we get $X \subseteq Y$ and $X \supseteq Y$ by Definition 1.7, and then $X = Y$ by Requirement 1.10. **Q.e.d. (Lemma 1.11)**

Definition 1.13 (Power Class, Meet Class)

$\mathfrak{P}(X) \;:=\; \{\, z \mid z \subseteq X \,\}$ "power class of X"

$\mathfrak{b}(X) \;:=\; \{\, z \mid \neg\mathcal{U}(z) \;\wedge\; \neg\mathcal{U}(X) \;\wedge\; z \cap X \neq \emptyset \,\}$ "meet class of X"

The symbol "$\mathfrak{b}$" is chosen for the meet class as in [Forster, 1995]. The reason for this is that "$\mathfrak{b}$" looks a little bit like an upside-down "$\mathfrak{P}$" and the two are dual as follows:

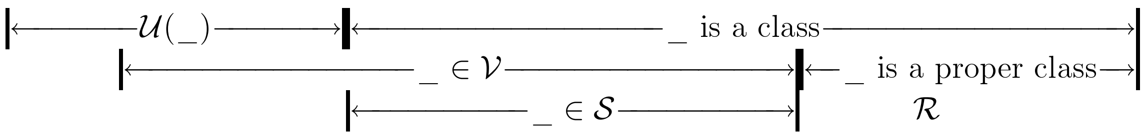
Corollary 1.14
$$\begin{aligned}\mathfrak{P}(X) &= \{\, z \mid \neg\mathcal{U}(z) \;\wedge\; \neg\mathcal{U}(X) \;\wedge\; \forall y \in z.\ (y \in X) \,\}\\ \mathfrak{b}(X) &= \{\, z \mid \neg\mathcal{U}(z) \;\wedge\; \neg\mathcal{U}(X) \;\wedge\; \exists y \in z.\ (y \in X) \,\}\end{aligned}$$

1.5 Sets and Classes

Definition 1.15 ([Proper] Class, Set)
X is a *class* if $\neg\mathcal{U}(X)$. X is a *proper class* if X is a class but not a set.
x is a *set* if $x \in \mathcal{S}$.

Not all classes can be sets: the Russell class $\mathcal{R}$ is the famous example. We have
$$\mathcal{R} \in \mathcal{R} \quad\Leftrightarrow\quad \mathcal{R} \notin \mathcal{R} \wedge \mathcal{R} \in \mathcal{V},$$
and thus:

Corollary 1.16 $\forall Z.\ (\mathcal{R} \notin Z) \quad\wedge\quad \mathcal{R} \notin \mathcal{V} \quad\wedge\quad \mathcal{R} \notin \mathcal{R} \quad\wedge\quad \neg\mathcal{U}(\mathcal{R})$.



Different class theories compete in getting $\mathcal{V}$ as big as possible. For example, in NF and ML we have $\mathcal{V} \in \mathcal{V}$. In NBG and in MK, however, a definable subclass of a set is always a set, which implies $\mathcal{V} \notin \mathcal{V}$.

Lemma 1.17 $\mathfrak{P}(\mathcal{V}) = \mathcal{S}$.

Proof of Lemma 1.17
The following are logically equivalent: $x \in \mathfrak{P}(\mathcal{V})$; $x \subseteq \mathcal{V} \wedge x \in \mathcal{V}$ (by Corollary 1.3 and Definition 1.13); $\neg\mathcal{U}(x) \wedge x \in \mathcal{V}$ (by Corollary 1.9); $x \in \mathcal{S}$ (by Corollary 1.3).
By Requirement 1.5, we have $\neg\mathcal{U}(\mathfrak{P}(\mathcal{V}))$ and $\neg\mathcal{U}(\mathcal{S})$.
Thus, an application of Lemma 1.11 completes the proof. **Q.e.d. (Lemma 1.17)**

As corollaries of Definition 1.15 as well as Corollary 1.9, Corollary 1.8, and Lemma 1.17, respectively, we get:

Corollary 1.18 $X \subseteq \mathcal{V}$ *iff* X *is a class.*

Corollary 1.19 $X \notin \mathcal{V} \wedge \neg\mathcal{U}(X)$ *iff* X *is a proper class.*

Corollary 1.20 $x \in \mathfrak{P}(\mathcal{V})$ *iff* x *is a set.*

1.6 Expansion of the Class Constructor

In § 1.1 we did not list the class constructor $\{\, x \mid A \,\}$ among the basic symbols of the signature of class theory. Thus, there must be a procedure to eliminate it from all formulas. And as formulas may be parts of terms (due to the class constructor and HILBERT's ε-operator), it is wise to eliminate it locally from all atomic predicates.

After a preprocessing phase, in which all symbols defined directly or indirectly in terms of the class constructor must be recursively replaced with their definitions, this elimination procedure applies the following rewrite steps:

Nested within a term: Replace all

atomic predicates $P[\{\, x \mid A \,\}]$ where
the class constructor $\{\, x \mid A \,\}$ occurs as an argument of the pair constructor

with

$$\exists Z.\ (Z = \{\, x \mid A \,\} \quad \wedge \quad P[Z])$$

for a new variable Z. For example, replace

$$(u, (v, \{\, x \mid A \,\})) \in w$$

with

$$\exists Z.\ (Z = \{\, x \mid A \,\} \quad \wedge \quad (u, (v, Z)) \in w).$$

To the left of $\in$: Replace all $\in$-atoms of the form

$$\{\, x \mid A \,\} \in t$$

with

$$\exists Z.\ (Z = \{\, x \mid A \,\} \quad \wedge \quad Z \in t)$$

for a new variable Z.

On both sides of $=$: Replace all $=$-atoms of the form

$$\{\, x \mid A \,\} = \{\, y \mid B \,\}$$

with

$$\forall z.\ (z \in \{\, x \mid A \,\} \quad \Leftrightarrow \quad z \in \{\, y \mid B \,\})$$

for a new variable z. This an equivalence transformation by Lemma 1.11 and Requirement 1.5.

On a single side of $=$: Replace all $=$-atoms of the forms

$\{\, x \mid A \,\} = t$ or $t = \{\, x \mid A \,\}$, where t is not a class constructor,

with

$$\neg\mathcal{U}(t) \ \wedge\ \forall z.\ (z \in \{\, x \mid A \,\} \quad \Leftrightarrow \quad z \in t)$$

for a new variable z. This an equivalence transformation by Lemma 1.11, Requirement 1.5, and Definition 1.6.

Urelements: Replace all atoms of the form

$$\mathcal{U}(\{\, x \mid A \,\})$$

with

$$\mathsf{false}.$$

This is an equivalence transformation according to Requirement 1.5.

To the right of $\in$: Replace all $\in$-atoms of the form

$$t \in \{\ x \mid A\ \} \quad \text{where the variable } x \text{ does not occur in the term } t$$

with

$$\exists x.\ (x = t \ \wedge\ A \ \wedge\ \exists Z.\ (x \in Z))$$

for a new variable Z. This is an equivalence transformation by Requirement 1.1. The quantification on the variable x is to avoid an increase of the number of occurrences of the term t — and especially of the formulas that t may contain — which might occur for the alternative of substituting t for x. In case that x does occur in t — to apply the replacement nevertheless — we have to rename the bound variable x in $\{\ x \mid A\ \}$ in advance.

Now any of these steps reduces the following measure w.r.t. the well-ordering which lexicographically combines the well-ordering of the natural numbers thrice:

1. Number of the occurrences of a class constructors at non-top-levels of terms or directly to the left of $\in$.
2. Number of the occurrences of $=$.
3. Number of the occurrences of class constructors.

Thus, all these steps together form a strongly terminating rewrite system in which all formulas containing class constructors are reducible.

1.7 Standard Axioms of Set Theory

While the class theories of QUINE's ML and NEUMANN–BERNAYS–GÖDEL are pro and contra "$\mathcal{V} \in \mathcal{V}$", i.e. pro and contra the universal class being a set, they agree on the axioms here, just as MORSE–KELLEY, ZERMELO–FRAENKEL, and QUINE's NF.

Requirement 1.21 (Axiom of the Empty Set and the Singleton Set)

$$\forall X.\ (\{X\} \in \mathcal{S}).$$

The title of Requirement 1.21 should become obvious from the following corollary of Corollary 1.16:

Corollary 1.22 $\{\mathcal{R}\} = \emptyset$.

Requirement 1.23 (Axiom of the Ordered Pair)

$$\forall x_1, y_1.\ \left(\begin{array}{c} (x_1, y_1) \in \mathcal{V} \\ \Leftrightarrow \left(\begin{array}{rl} & x_1 \in \mathcal{V} \\ \wedge & y_1 \in \mathcal{V} \end{array}\right) \end{array}\right) \quad \wedge \quad \forall x_1, x_2, y_1, y_2 \in \mathcal{V}.\ \left(\begin{array}{c} (x_1, y_1) = (x_2, y_2) \\ \Leftrightarrow \left(\begin{array}{rl} & x_1 = x_2 \\ \wedge & y_1 = y_2 \end{array}\right) \end{array}\right).$$

Requirement 1.24 (Axiom of Simple Operations)
$\forall x, y \in \mathcal{S}.\ (x \cup y,\ x \cap y,\ x \setminus y,\ {}_x{\upharpoonleft}\mathrm{id},\ x \times y,\ \mathrm{dom}(x),\ x^{-1},\ x \circ y \in \mathcal{S})$.

Corollary 1.25 $\forall x, y \in \mathcal{S}.\ (\mathrm{id}{\restriction}_x,\ \mathrm{ran}(x),\ \mathrm{field}(x),\ x{\restriction}_y,\ {}_y{\upharpoonleft}x,\ \langle y \rangle x,\ x \langle y \rangle \in \mathcal{S})$.

Note that the identity function "id" is not a set in ZF, NBG, and MK, but a set indeed in NF and ML. Moreover, note that, for conceptual reasons, we do not state $\forall x \in \mathcal{S}.\ (\mathfrak{P}(x) \in \mathcal{S})$ as an axiom here, although this is valid in all of ZF, NBG, MK, NF, and ML, but only in Definition 1.26 as a special axiom.

1.8 Rarely Needed Special Axioms of Set Theory

In rare occurrences, we will need the Power-Set Axiom, the Axiom of Collection (a strong form of the Axiom of Replacement, cf. e.g. [Aczel, 1988]) or the Principle of Dependent Choice (a weak form of the Axiom of Choice, cf. [Rubin & Rubin, 1985; Howard & Rubin, 1998]).

Definition 1.26 (Power-Set Axiom)

$$\forall x \in \mathcal{S}.\ (\ \mathfrak{P}(x) \in \mathcal{S}\ ).$$

Definition 1.27 (Axiom of Collection, [Jech, 2006, p. 65])
The *Axiom of Collection* is the following axiom scheme, where the variables X, Y must not occur in the formula A:

$$\forall X \in \mathcal{S}.\ \exists Y \in \mathcal{S}.\ \forall x \in X.\ (\ \exists y.\ A \quad \Rightarrow \quad \exists y \in Y.\ A\ )$$

1.9 The Critical Axiom of MK: Separation

The following axiom scheme is standard with ZF, NBG, and MK. It is critical insofar as it is inconsistent with the Axiom of Set Comprehension of QUINE's ML (cf. Definition 1.29), because the latter axiom implies $\mathcal{V}$ to be set and then the former axiom implies $\mathcal{R}$ to be a set as well.

Definition 1.28 (Axiom of Separation)
If A is a formula, x is a variable, and t is a term, then

$$t \in \mathcal{S} \quad \Rightarrow \quad \{\, x \in t \mid A \,\} \in \mathcal{S}.$$

1.10 The Critical Axiom of ML: Stratified Set Comprehension

For the very few parts of this text which strongly rely on QUINE's ML and QUINE's NF, we have to state ML's axiom scheme of *set* comprehension (cf. [Quine, 1981, p.159]), which — of course — is stricter than the one of *class* comprehension of Requirement 1.1. Moreover, while the latter one is common to ML and NBG, the former one is only part of ML and — after removing the restrictions to sets (i.e. elementship) — also of NF.

Definition 1.29 (Set Comprehension Axiom of QUINE's ML)
Let B be a stratified formula which contains no other free variables but $x, w_1, \ldots, w_n$.
Let A result from B by restricting all bound variables to membership, i.e. by replacing any subformula of the forms $\forall z.\ C$, $\exists z.\ C$, &c. with $\forall z \in \mathcal{V}.\ C$, $\exists z \in \mathcal{V}.\ C$, &c..
Then the *set comprehension axiom scheme of* QUINE*'s* ML says that:

$$w_1, \ldots, w_n \in \mathcal{V} \quad \Rightarrow \quad \{\, x \mid A \,\} \in \mathcal{S}.$$

Note that, in Definition 1.29, we may replace some of the "$z \in \mathcal{V}$" with "$z \in \mathcal{S}$" because this just means "$z \in \mathcal{V} \wedge \neg\mathcal{U}(z)$".

Definition 1.30 (Stratification, Stratifiedness)
N is an *integer-substituted formula* if there is a formula A such that N results from A by substituting all variables, bound as well as free ones, at all places, with integer numbers.
S is a *stratification* if S is can be rewritten to an integer-substituted formula S' where for each atom A occurring in S' there is some integer number n such that A is of one of the following atom patterns:

Urelement Predicate: $\mathcal{U}(n)$.

Element Predicate: $n \in (n{+}1)$.

Primitive Equality Predicate: $n = n$.

The rewriting may apply any rule of the following rewrite system from left to right:
Well-Typed Pair: $(n,n) \rightarrow n$.

HILBERT's ε: $\varepsilon n.\ S'' \rightarrow n$ if S'' is a stratification.

A formula A is *stratified* if for B resulting from A by expanding all defined symbols, including the class constructor according the procedure of § 1.6, and then renaming bound variables such that each bound variable is bound only once and does not occur free, there is a substitution σ from variables to integers, such that the application of σ to B, replacing both free and bound variables, at all places, is a stratification.

As expanding defined symbols is not always confluent in a strict sense, in Definition 1.30 we have to check all possible expansions in principle. The intention, however, is that it does not matter, which expansion we take. Moreover, expansion easily results in formulas of a size which humans cannot handle. Thus, instead of actually executing the expansion, we prefer to extend the confluent and strongly terminating rewrite system to handle definitions of terms and to extend the patterns to handle definitions of predicates:

Lemma 1.31 *If we extend the atom patterns and the rewrite system of Definition 1.30 as follows and do not enforce the definitional expansion of the respective symbols, this does not change the notion of stratifiedness:*

Subclass Predicate: $n \subseteq n$.

Functional Application: $(n{+}1)(n) \rightarrow n$.

Class Constructor: $\{\ n \mid S''\ \} \rightarrow (n{+}1)$ *if* S'' *is a stratification.*

Simple Operations:

$n{\cap}n \rightarrow n$	$\mathrm{dom}(n) \rightarrow n$	${}_{n}{\upharpoonleft}n \rightarrow n$
$n{\cup}n \rightarrow n$	$\mathrm{ran}(n) \rightarrow n$	$n{\upharpoonright}_{n} \rightarrow n$
$n{\setminus}n \rightarrow n$	$\mathrm{field}(n) \rightarrow n$	$\langle n\rangle n \rightarrow n$
$n{\times}n \rightarrow n$		$n\langle n\rangle \rightarrow n$
$n{\circ}n \rightarrow n$		

Power and Meet Class:

$\mathfrak{P}(n) \rightarrow (n{+}1)$	$\mathfrak{b}(n) \rightarrow (n{+}1)$

Proof of Lemma 1.31

As we will possibly change the whole definition of stratification in the further development, and as proper proofs would be very involved, we just sketch the proofs here.

Subclass Predicate: Substitute the new variable x of Definition 1.7 with $(n{-}1)$.

Functional Application: Consider the functional application $f(x)$. According to the definition of functional application in § 1.2, $f(x)$ expands to $\varepsilon y.\ ((x,y) \in f)$. Thus, the following are logically equivalent: there is some $n \in \mathbf{N}$ such that the definitional expansion of $f(x)$ instantiated according to $\sigma \uplus \{y{\mapsto}n\}$ can be rewritten with the old rewrite system to m; there is some $n \in \mathbf{N}$ such that $(\varepsilon y.\ ((x,y) \in f))(\sigma \uplus \{y{\mapsto}n\})$ can be rewritten with the old rewrite system to m; there is some $n \in \mathbf{N}$ such that $((x,y) \in f)(\sigma \uplus \{y{\mapsto}n\})$ is a stratification and $m = n$; $((x,y) \in f)(\sigma \uplus \{y{\mapsto}m\})$ is a stratification; $f\sigma = m{+}1$ and $x\sigma = m$; $(f(x))\sigma$ can be rewritten with the new rewrite system to m.

Class Constructor: Here we have to consider all the expansions of § 1.6. We assume exactly the notation of the respective cases in § 1.6.

"Nested within a term" or "To the left of $\in$": As, in the definitional expansion, $Z = \{\ x \mid A\ \}$ is added for a new variable Z, in case that the instantiated expansion is a stratification, $\{\ x \mid A\ \}\sigma$ must rewrite to $Z\sigma$, and $P[Z]\sigma$ must be a stratification. This is equivalent to the stratification without definitional expansion because $\{\ x \mid A\ \}$ now replaces Z in $P[Z]$. Note that the right-hand side of the new rewrite rule does not matter in this case. What matters is only the addition of a new rule with the given left-hand side.

"On both sides of =" or "On a single side of =": Trivial.

Urelements: Trivial.

To the right of $\in$: Assume that $(t \in \{\ x \mid A\ \})$ has passed definitional expansion with the exception of the given class constructor. Then the following are logically equivalent: there is an $n_Z \in \mathbf{N}$ such that, in the old rewrite system, definitional expansion of the class constructor in $(t \in \{\ x \mid A\ \})$ instantiated according to $\sigma \uplus \{Z \mapsto n_Z\}$ is a stratification; there is an $n_Z \in \mathbf{N}$ such that, in the old rewrite system,

$$\big(\ \exists x.\ (x = t\ \wedge\ A\ \wedge\ \exists Z.\ (x \in Z))\ \big)(\sigma \uplus \{Z \mapsto n_Z\})$$

is a stratification; there is an $n_Z \in \mathbf{N}$ such that, in the old rewrite system, $t\sigma$ rewrites to $x\sigma$, and $A\sigma$ is a stratification, and $n_Z = x\sigma{+}1$; in the old rewrite system, $t\sigma$ rewrites to $x\sigma$, and $A\sigma$ is a stratification; $(t \in \{\ x \mid A\ \})\sigma$ is a stratification in the new rewrite system. **Q.e.d. (Lemma 1.31)**

Example 1.32 (Stratifiedness, positive)
"$f(g(x,y)) \subseteq g(f(x),y)$" is stratified according to the substitution

$$\{x \mapsto n,\ \ y \mapsto n,\ \ f \mapsto (n{+}1),\ \ g \mapsto (n{+}1)\},$$

the application of which results first in $(n{+}1)((n{+}1)(n,n)) \subseteq (n{+}1)((n{+}1)(n),n)$,
then in $(n{+}1)((n{+}1)(n)) \subseteq (n{+}1)(n,n)$,
then in $(n{+}1)(n) \subseteq (n{+}1)(n)$,
and finally in $n \subseteq n$.

Example 1.33 (Stratifiedness, negative)

- "$x \notin x$" is not stratified, so that the Russell class may still be a proper class.
- Moreover, "$x \notin f(x)$" is not stratified, so that Cantor's 2nd diagonalization may produce a proper class.
- Finally, "α is full" (cf. Definition 3.7) is not stratified because its definiens
  $$\text{“}\forall x.\ ((x \in \alpha) \Rightarrow (x \subseteq \alpha))\text{”}$$
  is not stratified: Indeed, if we substitute x with n, to satisfy the atom pattern for "$x \subseteq \alpha$" of Lemma 1.31, we have to substitute α with n as well, resulting in
  $$\text{“}\forall n.\ ((n \in n) \Rightarrow (n \subseteq n))\text{”}$$
  containing the forbidden atom pattern $n \in n$, not permitted for a stratification according to Definition 1.30.

2 Basic Notions and Notation

Let '$\mathbf{N}$' denote the set of natural numbers and '$\prec$' the ordering on $\mathbf{N}$.

Let $\mathbf{N}_{+} := \{\, n \in \mathbf{N} \mid 0 \neq n \,\}$.

We use '$\uplus$' for the union of disjoint classes.

The *finite power-class operator* is defined as $\mathfrak{P}_{\mathbf{N}}(X) := \{\, x \mid x \subseteq X \;\wedge\; |x| \in \mathbf{N} \,\}$.

Let R be a binary relation. R is said to be a relation *on* A if $\mathrm{field}(R) \subseteq A$. R is *irreflexive* if $\mathrm{id} \cap R = \emptyset$. R is A-*reflexive* if ${}_{A}{\upharpoonleft}\mathrm{id} \subseteq R$. Speaking of a *reflexive* relation we refer to the largest A that is appropriate in the local context. And referring to this A we write R^0 to ambiguously denote ${}_{A}{\upharpoonleft}\mathrm{id}$. With $R^1 := R$, and $R^{n+1} := R^n \circ R$ for $n \in \mathbf{N}_{+}$, R^m denotes the m-step relation for R. R is *transitive* if $\forall x, y, z.\ \big(\ (x,y) \in R \wedge (y,z) \in R \ \Rightarrow\ (x,z) \in R\ \big)$. The *transitive closure* of R is $R^{+} := \bigcup_{n \in \mathbf{N}_{+}} R^n$. The *reflexive & transitive closure* of R is $R^{*} := \bigcup_{n \in \mathbf{N}} R^n$.

Furthermore, we use '$\emptyset$' to denote the empty set as well as the empty function. Functions are (right-) unique relations and the meaning of '$f \circ g$' is extensionally given by $(f \circ g)(x) = g(f(x))$. The *class of total functions from* A *to* B is denoted as $A \rightarrow B$. The *class of (possibly) partial functions from* A *to* B is denoted as $A \rightsquigarrow B$. Both $\rightarrow$ and $\rightsquigarrow$ associate to the right, i.e. $A \rightsquigarrow B \rightarrow C$ reads $A \rightsquigarrow (B \rightarrow C)$.

2.1 Suprema and Infima

Definition 2.1 (Dual, Upper Bound, Supremum)

Let $\leq$ be a binary relation (on A). Let P be a singulary predicate (on A).

The *dual of* $\leq$ is its reverse relation denoted by $\geq$.

s is *an $\leq$-upper bound of P* if

$$\forall x.\ (P(x) \Rightarrow x{\leq}s) \qquad (\mathrm{UB}^{\leq}(P,s))$$

s is *an $\leq$-supremum of P* if s is the least upper bound, i.e. an upper bound that is a *lower bound* (= dual of upper bound) of the upper bounds; formally:

$$\mathrm{UB}^{\leq}(P,s) \qquad (\sup 1)$$

and

$$\mathrm{UB}^{\geq}(\lambda u.\ \mathrm{UB}^{\leq}(P,u),\ s); \qquad (\sup 2)$$

or more explicitly:

$$\forall x.\ (P(x) \Rightarrow x{\leq}s) \qquad (\sup 1')$$

and

$$\forall x.\ \big(\ \forall u.\ (P(u) \Rightarrow u{\leq}x) \Rightarrow s{\leq}x\ \big). \qquad (\sup 2')$$

We denote such a supremum s with $\sup^{\leq}_{P(x)} x$, or simply with $\sup_{P(x)} x$.
An infimum is the dual of a supremum, $\inf^{\leq} = \sup^{\geq}$; more explicitly: s is *an $\leq$-infimum of P* if

$$\mathrm{UB}^{\geq}(P,s) \qquad (\inf 1)$$

and

$$\mathrm{UB}^{\leq}(\lambda u.\ \mathrm{UB}^{\geq}(P,u),\ s). \qquad (\inf 2)$$

For a class X, we write $\sup^{\leq} X$ for $\sup^{\leq}_{x\in X} x$.

Lemma 2.2 (Existence of Suprema = Existence of Infima)
If $\leq$ is a binary relation (on A) where any singulary predicate (on A) has a $\leq$-supremum (in A), then any predicate P (on A) has an $\leq$-infimum (in A), namely any $\leq$-supremum of $\lambda y.\ \mathrm{UB}^{\geq}(P,y)$ is an $\leq$-infimum of P; i.e., roughly speaking, we can always take

$$\inf^{\leq}_{P(x)} x \ :=\ \sup^{\leq}_{\mathrm{UB}^{\geq}(P,y)} y.$$

Proof of Lemma 2.2
As the second property (inf 2) of an $\leq$-infimum of P is identical to the first property (sup 1) of an $\leq$-supremum of $\lambda y.\ \mathrm{UB}^{\geq}(P,y)$, namely $(\sup 1)\{P \mapsto \lambda y.\ \mathrm{UB}^{\geq}(P,y)\}$, it suffices to show that the second property of the supremum, namely $(\sup 2)\{P{\mapsto}\lambda y.\ \mathrm{UB}^{\geq}(P,y)\}$, implies the first property of the infimum, i.e.

$$\mathrm{UB}^{\geq}(\lambda u.\ \mathrm{UB}^{\leq}(\lambda y.\ \mathrm{UB}^{\geq}(P,y),\ u),\ \ s) \ \ \Rightarrow\ \ \mathrm{UB}^{\geq}(P,s),$$

which is hardly comprehensible to humans, but most easily proved automatically.[4] Maybe humans would understand the following proof: “The class of upper bounds of the lower bounds of P is a super-class of the extension of P. Thus, any lower bound of the upper bounds of the lower bounds of P is a lower bound of P.” **Q.e.d. (Lemma 2.2)**

2.2 Well-Founded Orderings

Definition 2.3 ([$\mathcal{S}$-] Well-Founded, Total, Ordering)
Let $<$ be a binary relation. As always, let $>$, the *dual of* $<$, be defined by $a{>}b$ if $b{<}a$.
$<$ is [$\mathcal{S}$-] *well-founded* if for any class Q ($\subseteq \mathrm{field}(<)$) with $Q \neq \emptyset$ [and $Q \in \mathcal{S}$], we have

$$\exists m \in Q.\ \forall w \in Q.\ \neg(w{<}m).$$

$<$ is *total* if $<$ is total on field($<$). $<$ is *total on A* if $\forall a, b \in A.\ (a{<}b \vee a{>}b \vee a = b)$.
$<$ is an *ordering* [*on A*] if $<$ is an irreflexive and transitive relation [on A].

Note that, in the literature, an ordering is sometimes called "strict partial ordering" and a total ordering is often called a *linear ordering*.

Lemma 2.4 ([Wirth, 2004, Lemma 2.1])
For a binary relation R [with $R^+ \in \mathcal{S}$], we have the following logical equivalence:
R is [$\mathcal{S}$-] well-founded iff R^+ is an [$\mathcal{S}$-] well-founded ordering.

Proof of Lemma 2.4

The backward implication is trivial because R^+-minimality in a class Q implies R-minimality in Q due to $R \subseteq R^+$.

For the forward implication, since R^+ is clearly transitive, it suffices to show that it is [$\mathcal{S}$-] well-founded, because then it is irreflexive. Thus, suppose that there is some class Q with [$Q \in \mathcal{S}$ and] $\forall a \in Q.\ \exists a' \in Q.\ a' R^+ a$. We have to show that Q must be empty.

Set $B := \langle Q \rangle R^*$.

[As $B = Q \cup \langle Q \rangle R^+$, we get $B \in \mathcal{S}$ by Requirement 1.24 and Corollary 1.25.]

Claim 1: For any $b \in B$, there is some $b' \in B$ with $b'\ R\ b$.

Proof of Claim 1: Let us assume $b \in B$. Then, by definition of B and the property of Q, there are some $a, a' \in Q$ with $a'\ R^+\ a\ R^*\ b$. Thus, $a'\ R^+\ b$. Thus, there is some b' with $a'\ R^*\ b'\ R\ b$. Q.e.d. (Claim 1)

By Claim 1 and the assumption that R is [$\mathcal{S}$-] well-founded, we get $B = \emptyset$.
Then, we also have $Q = \emptyset$ due to $Q \subseteq B$. **Q.e.d. (Lemma 2.4)**

Although the following lemma will have only one direct application in this paper (namely in the Proof of Lemma 3.31), this application is crucial.

Lemma 2.5
If $<$ is an [$\mathcal{S}$-] well-founded relation and
if $f : \mathrm{field}(<) \rightarrow \mathrm{field}(<)$ satisfies $\forall a, b.\ \big(\ a < b \ \ \Rightarrow\ \ f(a) < f(b)\ \big)$;
then we have

$$\forall a.\ \neg(f(a) < a)$$

[*provided that either we assume the Set Comprehension Axiom of* ML *and* $<, f \in \mathcal{S}$, *or else we assume the Axiom of Separation and* $\mathrm{field}(<) \in \mathcal{S}$.]

Proof of Lemma 2.5

Set $Q := \{\, a \mid f(a) < a \,\}$.

[On the one hand, if we assume the Set Comprehension Axiom of ML (cf. Definition 1.29) and $<, f \in \mathcal{S}$, then we have $Q \in \mathcal{S}$. On the other hand, if we assume the Axiom of Separation (cf. Definition 1.28) and $\mathrm{field}(<) \in \mathcal{S}$, then due to $Q \subseteq \mathrm{field}(<)$, we have $Q \in \mathcal{S}$ as well.]

Assume that $Q \neq \emptyset$. Then, as $<$ is [$\mathcal{S}$-] well-founded, there is some $m \in Q$ with $\forall w \in Q.\ \neg(w < m)$. But then $f(m) < m$. Then $f(f(m)) < f(m)$. Thus, $f(m) \in Q$, contradicting $f(m) < m$. **Q.e.d. (Lemma 2.5)**

Lemma 2.6 *Let f be a function. Let $<_0$ be an ordering on* $\mathrm{dom}(f)$. *Let $<_1$ be an ordering on* $\mathrm{ran}(f)$. *If* $\forall a, b \in \mathrm{dom}(f).\ \big(\ a <_0 b \quad \Leftrightarrow \quad f(a) <_1 f(b)\ \big)$, *then*

$$\forall b \in \mathrm{dom}(f).\ \big(\ \langle {<_0}\langle\!\{b\}\!\rangle\rangle f \ = \ {<_1}\langle\!\{f(b)\}\!\rangle\ \big).$$

Proof of Lemma 2.6 The "$\subseteq$"-direction is trivial. To show the "$\supseteq$"-direction, assume $b \in \mathrm{dom}(f)$ and $c <_1 f(b)$. As $<_1$ is an ordering on $\mathrm{ran}(f)$, there is some $a \in \mathrm{field}(<_0)$ with $f(a) = c$. From $f(a) <_1 f(b)$, we get $a <_0 b$. Then $c = f(a) \in \langle {<_0}\langle\!\{b\}\!\rangle\rangle f$, as was to be shown. **Q.e.d. (Lemma 2.6)**

Definition 2.7 ([Ordering of a] [Well-Founded] Quasi-Ordering)
Let A be a class.
Let $\lesssim$ be a binary relation. As always, let $\gtrsim$, the *dual of* $\lesssim$, be defined by $a \gtrsim b$ if $b \lesssim a$.
$\lesssim$ is a *quasi-ordering on A* if $\lesssim$ is an A-reflexive and transitive relation on A.
The *ordering $<$ of a quasi-ordering* $\lesssim$ is $\lesssim \setminus \gtrsim$.
$\lesssim$ is an [$\mathcal{S}$-] *well-founded quasi-ordering* if $\lesssim$ is a quasi-ordering and $<$ is [$\mathcal{S}$-] well-founded.

Corollary 2.8 *The ordering of a quasi-ordering is an ordering.*

Definition 2.9 ([Ordering of a] [Well-Founded] Reflexive Ordering)
Let A be a class.
Let $\leq$ be a binary relation. As always, let $\geq$, the *dual of* $\leq$, be defined by $a \geq b$ if $b \leq a$.
$\leq$ is a *reflexive ordering on A* if $\leq$ is an anti-symmetric quasi-ordering on A.
$\leq$ is *anti-symmetric* if $\forall x, y.\ (x \leq y \wedge x \geq y \Rightarrow x = y)$.
The *ordering $<$ of a reflexive ordering* $\leq$ is the ordering of the quasi-ordering $\leq$.
$\leq$ is an [$\mathcal{S}$-] *well-founded reflexive ordering* if $\leq$ is an [$\mathcal{S}$-] well-founded quasi-ordering and a reflexive ordering.

Corollary 2.10 *The ordering $<$ of a reflexive ordering $\leq$ on A is exactly* $\leq \setminus ({}_A{\upharpoonleft}\mathrm{id})$.

Corollary 2.11 *If $<$ is an ordering on A, then* $< \cup\, {}_A{\upharpoonleft}\mathrm{id}$ *is a reflexive ordering on A.*

Corollary 2.12 *Let $\leq$ be a reflexive ordering on A. Let $<$ be the ordering of $\leq$. Then: $<$ is total on A iff $\leq$ is total [on A].*

Corollary 2.13
The dual of an [reflexive] [quasi-] ordering (on A) is an [reflexive] [quasi-] ordering (on A).

2.3 Order-Isomorphisms

Definition 2.14 ([$\mathcal{S}$-] Order-Isomorphism)
$\leq_0$ and $\leq_1$ are [$\mathcal{S}$-] *order-isomorphic* if there is an [$\mathcal{S}$-] order-isomorphism $f{::}\leq_0 \rightarrow \leq_1$.
$f{::}\leq_0 \rightarrow \leq_1$ is an [$\mathcal{S}$-] *order-isomorphism* if $\leq_0, \leq_1$ are reflexive orderings and
$f : \mathrm{field}(\leq_0) \rightarrow \mathrm{field}(\leq_1)$ is a bijection with $[f, \leq_0, \leq_1 \in \mathcal{S}$ and$]$

$$\forall a, b.\ (a \leq_0 b \ \Leftrightarrow\ f(a) \leq_1 f(b)).$$

As a corollary of Requirement 1.24 and Corollary 1.25 we get:

Corollary 2.15
The property of being [$\mathcal{S}$-] order-isomorphic is an equivalence on the reflexive orderings $\leq$ [with $\mathrm{field}(\leq) \in \mathcal{S}$]; i.e. it is reflexive (due to $({}_{\mathrm{field}(\leq)}{\upharpoonleft}\mathrm{id}){::}\leq \rightarrow \leq$), symmetric, and transitive.

Corollary 2.16
Let $\leq_0, \leq_1$ be reflexive orderings. Let $<_i$ being the ordering of $\leq_i$ $(i \in \{0, 1\})$.
Let $f : \mathrm{field}(\leq_0) \rightarrow \mathrm{field}(\leq_1)$ be a bijection.
Now the following two items are logically equivalent:

1. $f{::}\leq_0 \rightarrow \leq_1$ *is an* order-isomorphism.
2. $\forall a, b.\ (a <_0 b \ \Leftrightarrow\ f(a) <_1 f(b))$.

Moreover, if $\leq_0$ is total, then already the following is logically equivalent:

3. $\forall a, b.\ (a <_0 b \ \Rightarrow\ f(a) <_1 f(b))$.

As the concept of an ordering is simpler than the concept of a reflexive ordering, but somehow equivalent according to Corollaries 2.10 and 2.11, the following question may arise from Corollary 2.16: Why do we not use orderings instead of reflexive orderings in Definition 2.14? The answer is simple: If we took $\mathrm{field}(<_i)$ instead of $\mathrm{field}(\leq_i)$, we could not refer anymore to those elements of $\mathrm{field}(\leq_i)$ that are not connected with other elements via $<_i$. And the possibility of such a reference will become important in § 3.1.

Lemma 2.17
If $f{::}\leq_0 \rightarrow \leq_1$ is an order-isomorphism, then the following three are logically equivalent:

(i) $f{::}\leq_0 \rightarrow \leq_1$ *is an $\mathcal{S}$-order-isomorphism.*

(ii) $f, \leq_0 \in \mathcal{S}$.

(iii) $f, \leq_1 \in \mathcal{S}$.

Proof of Lemma 2.17
We have $\leq_1 = f^{-1} \circ \leq_0 \circ f$ and $\leq_0 = f \circ \leq_1 \circ f^{-1}$. Thus, everything is clear from the Axiom of Simple Operations (cf. Requirement 1.24). **Q.e.d. (Lemma 2.17)**

2.4 Initial Segments

Definition 2.18 (Initial Segment)
Let $\leq$ be a reflexive ordering. Let $<$ be the ordering of $\leq$.
$\leq'$ is an (proper) *initial segment of* $\leq$ if there is an $a \in \mathrm{field}(\leq)$ such that $\leq'$ is the initial segment of $\leq$ below a.
The *initial segment of* $\leq$ *below* a is ${}_A{\upharpoonleft}{\leq}{\upharpoonright}_A$ for $A := <\langle\!\{a\}\!\rangle$, and we denote it by $\lfloor\leq\rfloor_a$.

Lemma 2.19 *Let $\lfloor\leq\rfloor_a$ be an initial segment of a reflexive ordering $\leq$ on A. Then $\lfloor\leq\rfloor_a$ is a reflexive ordering on a proper subclass of A. Moreover, if $\leq\,\in\mathcal{S}$, then $\lfloor\leq\rfloor_a\in\mathcal{S}$.*

Proof of Lemma 2.19 If $\lfloor\leq\rfloor_a$ is an initial segment of $\leq\,\in\mathcal{S}$, then $a\in\mathrm{field}(\leq)$. By the Axiom of the Singleton Set (cf. Requirement 1.21), we have $\{a\}\in\mathcal{S}$. By Corollary 1.25, we have $\mathrm{field}(\leq)\in\mathcal{S}$, and then by the Axiom of Simple Operations (cf. Requirement 1.24) $({}_{\mathrm{field}(\leq)}{\upharpoonleft}\mathrm{id})\in\mathcal{S}$, and then, by Corollary 2.10, $< \; = \; \leq\setminus({}_{\mathrm{field}(\leq)}{\upharpoonleft}\mathrm{id}) \in \mathcal{S}$. Then, still by the Axiom of Simple Operations and Corollary 1.25, $A := <\langle\!\{a\}\!\rangle \in \mathcal{S}$, and $\lfloor\leq\rfloor_a = {}_A{\upharpoonleft}{\leq}{\upharpoonright}_A \in \mathcal{S}$. **Q.e.d. (Lemma 2.19)**

Lemma 2.20 *Let $f{::}{\leq_0}{\rightarrow}{\leq_1}$ be an order-isomorphism.*
Let $<_i$ be the ordering of $\leq_i$ $(i\in\{0,1\})$.
Then $\forall b \in \mathrm{field}(\leq_0).\ \big(\ \langle<_0\langle\!\{b\}\!\rangle\rangle f \;=\; <_1\langle\!\{f(b)\}\!\rangle\ \big)$.

Proof of Lemma 2.20 By Corollary 2.16, we get $\forall a, b.\ (a <_0 b \;\Leftrightarrow\; f(a) <_1 f(b))$. Then, by Lemma 2.6, we get $\forall b \in \mathrm{dom}(f).\ \big(\ \langle<_0\langle\!\{b\}\!\rangle\rangle f \;=\; <_1\langle\!\{f(b)\}\!\rangle\ \big)$.
Q.e.d. (Lemma 2.20)

Lemma 2.21 *If $f_1{::}{\leq_0}{\rightarrow}\lfloor\leq_1\rfloor_{a_1}$ and $f_2{::}{\leq_1}{\rightarrow}{\leq_2}$ are [$\mathcal{S}$-] order-isomorphisms, then $(f_1{\circ}f_2){::}{\leq_0}{\rightarrow}\lfloor\leq_2\rfloor_{f_2(a_1)}$ is an [$\mathcal{S}$-] order-isomorphism as well.*

Proof of Lemma 2.21
By Lemma 2.20 it is obvious that $f_2'{::}\lfloor\leq_1\rfloor_{a_1}{\rightarrow}\lfloor\leq_2\rfloor_{f_2(a_1)}$ is an order-isomorphism for $f_2' := {}_{\mathrm{field}(\lfloor\leq_1\rfloor_{a_1})}{\upharpoonleft}f_2$. As $f_1 \circ f_2 = f_1 \circ f_2'$, we know that $(f_1{\circ}f_2){::}{\leq_0}{\rightarrow}\lfloor\leq_2\rfloor_{f_2(a_1)}$ is an order-isomorphism by Corollary 2.15 [and also an $\mathcal{S}$-order-isomorphism by the Axiom of Simple Operations (cf. Requirement 1.24) and Lemma 2.17]. **Q.e.d. (Lemma 2.21)**

2.5 Order-Types

Definition 2.22 (Order-Type “$\mathcal{OT}(\leq, X)$”)
Let X be any class. Let $\leq$ be a reflexive ordering.
The the *order-type of* $\leq$ *over* X is

$$\mathcal{OT}(\leq, X) := \left\{ \ \leq' \ \middle| \ \begin{array}{ll} & \leq' \text{ is } \mathcal{S}\text{-order-isomorphic to } \leq \\ \wedge & \mathrm{field}(\leq') \subseteq X \end{array} \right\}.$$

As a corollary of Corollaries 1.25 and 2.15 we get:

Corollary 2.23
If $\leq \in \mathcal{S}$ *is a reflexive ordering with* $\mathrm{field}(\leq) \subseteq X$, *then* $\leq \ \in \ \mathcal{OT}(\leq, X)$.

The following lemma will find its applications in § 3.5, especially in Corollary 3.37 and Lemma 3.43.

Lemma 2.24 *If* $X \in \mathcal{S}$ *and if* $\leq \in \mathcal{S}$ *is a reflexive ordering, then*

$$\mathcal{OT}(\leq, X) \ \in \ \mathcal{S},$$

provided that we assume either the Set Comprehension Axiom of ML, *or else both the Power-Set Axiom and the Axiom of Separation.*

Proof of Lemma 2.24

According to Definition 2.14, $\leq'$ and $\leq$ are $\mathcal{S}$-order-isomorphic if $\leq', \leq \in \mathcal{S}$ are reflexive orderings and

$$\exists f \in \mathcal{S}. \left(\begin{array}{ll} & \mathrm{dom}(f) = \mathrm{field}(\leq') \\ \wedge & \mathrm{ran}(f) = \mathrm{field}(\leq) \\ \wedge & \forall (a', a), (b', b) \in f. \left(\begin{array}{ll} & (a' = b' \ \Leftrightarrow \ a = b) \\ \wedge & (a' \leq' b' \ \Leftrightarrow \ a \leq b) \end{array} \right) \end{array} \right).$$

In case we assume the Set Comprehension Axiom of ML (cf. Definition 1.29), we just have to show that the defining statement of Definition 2.22 is stratified and equivalent to the version where all quantifiers are restricted to elementship. Stratification is easy: Just set $f, \leq', \leq, X$ to the same integer number. $f, \leq', \leq, X$ are explicitly restricted to elementship. And we have $\mathrm{field}(\leq'), \mathrm{field}(\leq) \in \mathcal{S}$ by Corollary 1.25. The bound variables introduced by definitional expansion are all restricted to elementship in $\mathrm{field}(\leq')$ or $\mathrm{field}(\leq)$, either directly or as an element of a pair in f, $\leq'$, or $\leq$, which is equivalent due to the Axiom of the Ordered Pair (cf. Requirement 1.23).

In case we assume the Power-Set Axiom (cf. Definition 1.26) and the Axiom of Separation (cf. Definition 1.28), we have to find a $t \in \mathcal{S}$ such that $\mathcal{OT}(\leq, X) \subseteq t$. However, $t := \mathfrak{P}(X \times X)$ will do: On the one hand, according to the Axiom of Simple Operations (cf. Requirement 1.24) and the assumed Axiom of Power-Set, t is a set. On the other hand, from $\mathrm{field}(\leq') \subseteq X$ we indeed get $\mathcal{OT}(\leq, X) \subseteq t$, by the Axiom of the Ordered Pair (cf. Requirement 1.23). **Q.e.d. (Lemma 2.24)**

3 Well-Orderings

3.1 The Most Basic Concept of Well-Ordering

Definition 3.1 ([Irreflexive] [$\mathcal{S}$-] Well-Ordering)
$<$ is an *irreflexive* [$\mathcal{S}$-] *well-ordering on* A if $<$ is an ordering on A and
for every class $Q \subseteq A$ with $Q \neq \emptyset$ [and $Q \in \mathcal{S}$], we have $\exists m \in Q.\ \forall w \in Q \setminus \{m\}.\ (m<w)$.
$\leq$ is a [$\mathcal{S}$-] *well-ordering on* A if $\leq$ is a reflexive ordering on A and
for every class $Q \subseteq A$ with $Q \neq \emptyset$ [and $Q \in \mathcal{S}$], we have $\exists m \in Q.\ \forall w \in Q.\quad (m \leq w)$.

Note that it is a well-justified standard to define a well-ordering as a reflexive ordering instead of an (irreflexive) ordering, although the latter is the simpler concept in general. The reason is not that we save "$\setminus\{m\}$", but that we want to have $n+1$ well-ordered order-types over a set with n elements, for each $n \in \mathbf{N}$. (More generally, we want HARTOGS' Ordinal Theorem to hold; cf. Theorem 3.46.)

Example 3.2 Let us take $n := 2$. Considered as reflexive orderings we have the 3 well-ordered order-types of $\emptyset$, $\{(0,0)\}$, and $\{(0,0),(0,1),(1,1)\}$. We want to represent these 3 order-types by the 3 *ordinals* 0 (or $\emptyset$), 1 (or $\{0\}$), and 2 (or $\{0,1\}$). Considered as irreflexive well-orderings, however, we get only the 2 well-ordered order-types of $\emptyset$ and $\{(0,1)\}$, because the operation of taking the ordering of a reflexive ordering maps both $\emptyset$ and $\{(0,0)\}$ to the same element $\emptyset$.

Moreover, note that the reference of an irreflexive well-ordering to a field A is necessary because $\emptyset$ is an irreflexive well-ordering on $\emptyset$ and on $\{0\}$, but not on $\{0,1\}$.

Regarding "$\forall w \in Q \setminus \{m\}.\ (m<w)$" in the definition of an irreflexive well-ordering, the restriction of "$Q \subseteq A$" is necessary. Note that the restriction of "$Q \subseteq \mathrm{field}(<)$" is irrelevant, however, in the definition of well-foundedness (cf. Definition 2.3), because each $m \in Q \setminus \mathrm{field}(<)$ satisfies $\forall w \in Q.\ \neg(w<m)$ trivially.

All in all — regarding well-orderings — reflexive orderings are more convenient than (irreflexive) orderings.

Lemma 3.3 *Let $<$, $\leq$, and R be binary relations.*

1. *The following are logically equivalent:*
   - *(a)* $\leq$ *is an* $[\mathcal{S}\text{-}]$ *well-ordering on* A.
   - *(b)* $\leq$ *is a total and* $[\mathcal{S}\text{-}]$ *well-founded reflexive ordering on* A.
   - *(c)* $\leq$ *is an anti-symmetric relation on* A *and for every class* $Q \subseteq A$ *with* $Q \neq \emptyset$ $[$*and* $Q \in \mathcal{S}]$, *we have* $\exists m \in Q.\ \forall w \in Q.\ (m \leq w)$.
2. *The following are logically equivalent:*
   - *(a)* $<$ *is an irreflexive* $[\mathcal{S}\text{-}]$ *well-ordering on* A.
   - *(b)* $<$ *is a transitive relation on* A *which is total on* A *and* $[\mathcal{S}\text{-}]$ *well-founded.*
3. *The following are logically equivalent:*
   - *(a)* $R \cup {}_A{\upharpoonright}\mathrm{id}$ *is an* $[\mathcal{S}\text{-}]$ *well-ordering on* A
   - *(b)* $R \setminus ({}_A{\upharpoonright}\mathrm{id})$ *is an irreflexive* $[\mathcal{S}\text{-}]$ *well-ordering on* A.
4. *If* $\leq$ *is a reflexive ordering on* A *and* $<$ *is the ordering of* $\leq$, *then we have:* $\leq$ *is an* $[\mathcal{S}\text{-}]$ *well-ordering on* A *iff* $<$ *is an irreflexive* $[\mathcal{S}\text{-}]$ *well-ordering on* A.

Note that, for a well-ordering, in addition to $\exists m \in Q.\ \forall w \in Q.\ (m \leq w)$, anti-symmetry must be required because $\{(0,0),(0,1),(1,0),(1,1)\}$ is transitive and reflexive, but not a well-ordering. Reflexivity and transitivity, however, are redundant, as shown in Item (1c) of Lemma 3.3.

For an irreflexive well-ordering, however, in addition to $\exists m \in Q.\ \forall w \in Q \setminus \{m\}.\ (m < w)$, irreflexivity must be required because $\{(0,0),(0,1),(1,0),(1,1)\}$ is transitive, but not an irreflexive well-ordering, and transitivity must be required because $\{(0,1),(1,0)\}$ is irreflexive, but not an irreflexive well-ordering. For this reason, Lemma 3.3 has no Item (2c) analogous to Item (1c).

Proof of Lemma 3.3

"(1a)⇒(1b)": Let $\leq$ be an $[\mathcal{S}\text{-}]$ well-ordering on A. Then $\leq$ is a reflexive ordering on A. Moreover, for any $Q \subseteq \mathrm{field}(\leq)$ with $Q \neq \emptyset$ $[$and $Q \in \mathcal{S}]$, there is some $m \in Q$ such that $\forall w \in Q.\ (m \leq w)$. Let $<$ be the ordering of $\leq$. Let us show that $<$ is $[\mathcal{S}\text{-}]$ well-founded. If there were some $v \in Q$ with $v < m$, then $v \in Q \setminus \{m\}$, as $<$ is irreflexive. But then $v \leq m \leq v$ contradicts anti-symmetry of $\leq$. To show that $\leq$ is total, suppose $a, b \in \mathrm{field}(\leq)$. Setting Q to $\{a, b\}$, we get $a \leq b \vee b \leq a$ [by the Axiom of the Singleton Set (cf. Requirement 1.21) and the Axiom of Simple Operations (cf. Requirement 1.24)].

"(1b)⇒(1c)": Let $\leq$ be a total and $[\mathcal{S}\text{-}]$ well-founded reflexive ordering on A. Let $<$ be the ordering of $\leq$. Then, for any Q with $Q \neq \emptyset$, $[Q \in \mathcal{S}]$, and $Q \subseteq A$, there is some $m \in Q$ such that $\forall w \in Q.\ \neg(w < m)$. We have to show $m \leq w$ for each $w \in Q$. For deriving a contradiction, suppose $\neg(m \leq w)$ for some $w \in Q$. Then, as $\leq$ is A-reflexive, we have $m \neq w$. Then, as $\leq$ is total and A-reflexive, we have $m \geq w$. Then, as $\neg(m \leq w)$, we have $m > w$, contradicting $\forall w \in Q.\ \neg(w < m)$.

<u>"(1c)⇒(1a)":</u> Setting Q to the respective singleton set, we get that $\leq$ is A-reflexive [by the Axiom of Singleton Set (cf. Requirement 1.21)]. Then $\mathrm{field}(\leq) = A$. All we have left to show is transitivity. Thus, let us assume $a{\leq}b{\leq}c$. Set $Q := \{a, b, c\}$ [which is a set by the Axiom of the Singleton Set (cf. Requirement 1.21) and the Axiom of Simple Operations (cf. Requirement 1.24)]. Then, we get an $m \in Q$ such that $\forall w \in Q.\ (m{\leq}w)$. If $m{=}a$, then we get $a{\leq}c$ as required. If $m{=}b$, then we get $b{\leq}a$ and by anti-symmetry $a{=}b$, i.e. again the required $a{\leq}c$. Finally, if $m{=}c$, we get $c{\leq}a$ and by anti-symmetry $a{=}c$, and by A-reflexivity again the required $a{\leq}c$.

<u>"(2a)⇒(2b)":</u> Let $<$ be an irreflexive [$\mathcal{S}$-] well-ordering on A. Then $<$ is an ordering on A, i.e. an irreflexive and transitive relation on A. Moreover, for any $Q \subseteq A$ with $Q{\neq}\emptyset$ [and $Q{\in}\mathcal{S}$], there is some $m \in Q$ such that $\forall w \in Q{\setminus}\{m\}.\ (m{<}w)$.
Let us show that $<$ is [$\mathcal{S}$-] well-founded. If there were some $v \in Q$ with $v{<}m$, then $v \in Q{\setminus}\{m\}$, as $<$ is irreflexive. Thus, $v{<}m{<}v$, which contradicts $<$ being an ordering.
To show that $<$ is total on A, suppose $a, b \in A$. Setting Q to $\{a, b\}$ [which is a set by the Axiom of the Singleton Set (cf. Requirement 1.21) and the Axiom of Simple Operations (cf. Requirement 1.24)], we get $a{<}b \vee b{<}a \vee a{=}b$.

<u>"(2b)⇒(2a)":</u> Let $<$ be a transitive relation on A which is total on A and [$\mathcal{S}$-] well-founded. Then, for any $Q \subseteq A$ with $Q{\neq}\emptyset$ [and $Q{\in}\mathcal{S}$], there is some $m \in Q$ such that $\forall w \in Q.\ \neg(w{<}m)$.
Setting Q to the respective singleton set [by the Axiom of Singleton Set (cf. Requirement 1.21)], we see that $<$ is irreflexive, i.e. an ordering.
As $<$ is total on A, we have $\forall w \in Q{\setminus}\{m\}.\ (m{<}w)$. Thus, $\forall w \in Q.\ (m{\leq}w)$.

<u>"(3a)⇒(3b)":</u> Let us assume that $\leq\ :=\ R \cup {}_A{\upharpoonleft}\mathrm{id}$ is an [$\mathcal{S}$-] well-ordering on A. Then, by Corollary 2.10, $R \setminus ({}_A{\upharpoonleft}\mathrm{id})$ is exactly the ordering $<$ of $\leq$. By (1b), $<$ is [$\mathcal{S}$-] well-founded. By (2), it suffices to show that $<$ is total on A. But this is the case by Corollary 2.12, because $\leq$ is total by (1b).

<u>"(3b)⇐(3a)":</u> Let us assume that $<\ :=\ R \setminus ({}_A{\upharpoonleft}\mathrm{id})$ is an irreflexive [$\mathcal{S}$-] well-ordering on A. By Corollary 2.11, $\leq\ :=\ R \cup {}_A{\upharpoonleft}\mathrm{id}$ is a reflexive ordering on A. By (2), $<$ is total and [$\mathcal{S}$-] well-founded. Thus, by Corollary 2.12, $\leq$ is a total and [$\mathcal{S}$-] well-founded reflexive ordering on A, i.e. (1b) holds. Thus, (1a) holds as well.

<u>(4):</u> By (3) and Corollaries 2.10 and 2.11. **Q.e.d. (Lemma 3.3)**

Corollary 3.4 (min)
Let $\leq$ be an [$\mathcal{S}$-] well-ordering. Let $<$ be the ordering of $\leq$.
For any $Q \subseteq \mathrm{field}(\leq)$ with $Q{\neq}\emptyset$ [and $Q{\in}\mathcal{S}$],
there is a <u>unique</u> $m \in Q$ with $\forall w \in Q.\ (m{\leq}w)$,
and we denote this m by $\min_{\leq} Q$ or by $\min_{<} Q$.

Definition 3.5 ($\leq$-Successor $\mathrm{S}_{\leq}$, Limit Point)
Let $\leq$ be an [$\mathcal{S}$-] well-ordering. Let $<$ be the ordering of $\leq$. Let $a, b \in \mathrm{field}(\leq)$.
[Assume either the Set Comprehension Axiom of ML and $< \in \mathcal{S}$, or else the Axiom of Separation and $\mathrm{ran}(<) \in \mathcal{S}$.]
If $\exists c.\ (a{<}c)$, then the *$\leq$-successor of a* is $\mathrm{S}_{\leq}(a) := \min_{\leq}\{\ c \mid a{<}c\ \}$.
b is a *non-limit $\leq$-point* if there is some $a \in A$ such that $b = \mathrm{S}_{\leq}(a)$.
b is a *limit $\leq$-point* if b is not a non-limit $\leq$-point.

Lemma 3.6 ($\leq$-Predecessor)
Let $\leq$ be an [$\mathcal{S}$-] well-ordering on A.
[Assume either the Set Comprehension Axiom of ML *and* $< \in \mathcal{S}$, *or else the Axiom of Separation and* $\mathrm{ran}(<) \in \mathcal{S}$.]
Then, for every non-limit $\leq$-point $b \in A$, there is a <u>unique</u> $a \in A$ such that $b = \mathrm{S}_{\leq}(a)$.
This a is called the <u>$\leq$-predecessor of b</u>.

Proof of Lemma 3.6
Suppose that $\mathrm{S}_{\leq}(a) = \mathrm{S}_{\leq}(a')$. We have to show $a = a'$. Let $<$ be the ordering of $\leq$. As $<$ is total by Lemma 3.3, due to symmetry in a and a', it suffices to refute $a{<}a'$. But in the latter case we have $\mathrm{S}_{\leq}(a) \leq a' < \mathrm{S}_{\leq}(a')$, which contradicts $\mathrm{S}_{\leq}(a) = \mathrm{S}_{\leq}(a')$.

Q.e.d. (Lemma 3.6)

3.2 NEUMANN Ordinals

The implementation of ordinal numbers resulting from the following definition is essentially due to JOHN VON NEUMANN (1903–1957).

Definition 3.7 (Fullness, $\in_\alpha$, NEUMANN Ordinal)
α is *full* if $\neg\mathcal{U}(\alpha) \quad \wedge \quad \forall x\in\alpha.\ (x\subseteq\alpha)$.
The relation $\in_\alpha$ is given as $\{\ (x,y) \mid \alpha\ni x\in y\in\alpha\ \}$.
α is an $[\mathcal{S}$-$]$ NEUMANN *ordinal* if α is full and $\in_\alpha$ is an irreflexive $[\mathcal{S}$-$]$ well-ordering on α.

Corollary 3.8 *If α is a* NEUMANN *ordinal, then α is an $\mathcal{S}$-*NEUMANN *ordinal as well.*

The converse of Corollary 3.8 seems to require some extra presuppositions, but it is too early to discuss this question now; cf. Lemma 3.18.

In the literature, "transitive class" is sometimes used instead of "full", but we want to reserve that name for its standard meaning referring to binary relations.

Note that the class of $[\mathcal{S}$-$]$ NEUMANN ordinals cannot be defined via the *set* comprehension scheme of QUINE's NF or QUINE's ML (cf. Definition 1.29) because neither the notion of fullness (cf. Example 1.33) nor the relation $\in_\alpha$ are stratified, cf. e.g. [Holmes, 1998]. And indeed, the class of $[\mathcal{S}$-$]$ NEUMANN ordinals cannot be a set, cf. Corollary 3.25.

We will show here that the NEUMANN ordinals can be defined and understood without fixing a special theory of sets and classes such as ZF, NBG, MK, or QUINE's ML, and without any axioms, but the standard axioms of § 1.7. Especially, no axioms of choice, foundation, infinity, set comprehension, separation, subset, or power-set are required.

Corollary 3.9 *If α is an $[\mathcal{S}$-$]$* NEUMANN *ordinal, then $\alpha\notin\alpha$.*

Corollary 3.10 *$\emptyset$ is an $[\mathcal{S}$-$]$* NEUMANN *ordinal.*

Corollary 3.11
If α and β are $[\mathcal{S}$-$]$ NEUMANN *ordinals, then $\alpha\cap\beta$ is an $[\mathcal{S}$-$]$* NEUMANN *ordinal as well.*

Corollary 3.12
If $\beta\in\alpha$ and α is full, then $\forall\gamma.\ (\gamma\in\beta \quad\Leftrightarrow\quad \gamma\in_\alpha\beta)$.

Lemma 3.13 *If $\beta\in\alpha$ and α is an $[\mathcal{S}$-$]$* NEUMANN *ordinal, then $\beta = {\in_\alpha}\langle\{\beta\}\rangle$ is an $[\mathcal{S}$-$]$* NEUMANN *ordinal, too.*

Proof of Lemma 3.13
As α is full and $\beta\in\alpha$, we have $\beta\subseteq\alpha$.
$\beta = {\in_\alpha}\langle\{\beta\}\rangle$ already follows from Corollary 3.12 and Requirement 1.10.
Then $\in_\beta \;=\; {}_\beta{\upharpoonleft}{\in_\alpha}{\upharpoonright}_\beta$. Thus, as $\in_\alpha$ is an irreflexive $[\mathcal{S}$-$]$ well-ordering on α, $\in_\beta$ is an irreflexive $[\mathcal{S}$-$]$ well-ordering on β.
It now suffices to show that β is full, and for this (as $\beta\subseteq\alpha$ implies $\neg\mathcal{U}(\beta)$), it suffices to show $\eta\in\beta$ for $\eta\in\xi\in\beta$. Thus, let us suppose the latter. As α is full, we have first $\eta\in\xi\in\alpha$, and then also $\eta\in\alpha$. Thus, we have $\eta\in_\alpha\xi\in_\alpha\beta$. As $\in_\alpha$ is transitive, we have $\eta\in_\alpha\beta$, i.e. $\eta\in\beta$, as was to be shown. **Q.e.d. (Lemma 3.13)**

Lemma 3.14 *If α is an [$\mathcal{S}$-] NEUMANN ordinal, then $\alpha \cup \{\alpha\}$ is an [$\mathcal{S}$-] NEUMANN ordinal, too.*

Be warned that in case of $\alpha \notin \mathcal{V}$, we have $\alpha \cup \{\alpha\} = \alpha$.

Proof of Lemma 3.14

Let α be a [$\mathcal{S}$-] NEUMANN ordinal.

In case of $\alpha \notin \mathcal{S}$, we have $\alpha \cup \{\alpha\} = \alpha$ and the lemma is trivial. Thus, in the following, we may assume $\alpha \in \mathcal{S}$.

$\underline{\alpha \cup \{\alpha\} \text{ is full:}}$ If $x \in \alpha \cup \{\alpha\}$, then $x \in \alpha$ or $x = \alpha$. As α is full, this means $x \subseteq \alpha$ in both cases.

$\underline{\in_{\alpha \cup \{\alpha\}} \text{ is irreflexive:}}$ Suppose $\beta \in_{\alpha \cup \{\alpha\}} \beta$. In case of $\beta \in \alpha$, we get $\beta \in_\alpha \beta$, contradicting that $\in_\alpha$ is an irreflexive ordering. Otherwise, in case of $\beta = \alpha$, we get $\alpha \in \alpha$, contradicting Corollary 3.9.

$\underline{\in_{\alpha \cup \{\alpha\}} \text{ is transitive:}}$ Suppose $\eta \in_{\alpha \cup \{\alpha\}} \xi \in_{\alpha \cup \{\alpha\}} \beta$. In case of $\beta \in \alpha$, as α is full, we have $\eta \in_\alpha \xi \in_\alpha \beta$ and then, as $\in_\alpha$ is transitive, $\eta \in_\alpha \beta$, i.e. $\eta \in_{\alpha \cup \{\alpha\}} \beta$. Otherwise, in case of $\beta = \alpha$, as α is full, we have $\eta \in \alpha = \beta$, and then $\eta \in_{\alpha \cup \{\alpha\}} \beta$, again.

$\underline{\in_{\alpha \cup \{\alpha\}} \text{ is an irreflexive [}\mathcal{S}\text{-] well-ordering on } \alpha \cup \{\alpha\}\text{:}}$ Suppose $Q \subseteq \alpha \cup \{\alpha\}$ with $Q \neq \emptyset$ [and $Q \in \mathcal{S}$]. In case of $Q = \{\alpha\}$, we have $\min_{\in_{\alpha \cup \{\alpha\}}} Q = \alpha$. Otherwise, we set $Q' := Q \cap \alpha$. Then $\emptyset \neq Q' \subseteq \alpha$. [Moreover, from $\alpha \in \mathcal{S}$, we get $Q' \in \mathcal{S}$ due to the Axiom of Simple Operations (cf. Requirement 1.24).] As α is an [$\mathcal{S}$-] NEUMANN ordinal, we have $\min_{\in_\alpha} Q' \in \alpha$ by Corollary 3.4, i.e. $\min_{\in_{\alpha \cup \{\alpha\}}} Q = \min_{\in_\alpha} Q'$. **Q.e.d. (Lemma 3.14)**

Lemma 3.15 *If α is an [$\mathcal{S}$-] NEUMANN ordinal, β is full, and $\beta \subsetneq \alpha$ [and $\alpha \backslash \beta \in \mathcal{S}$], then $\beta = \min_{\in_\alpha} (\alpha \backslash \beta) \in \alpha$.*

I do not know how to show $\beta \in \alpha$ for an $\mathcal{S}$-NEUMANN ordinal α and a full $\beta \subsetneq \alpha$, unless $\alpha \backslash \beta$ is required to be a set.

Proof of Lemma 3.15

Due to $\beta \subsetneq \alpha$, we have $\alpha \backslash \beta \neq \emptyset$. Thus, $\min_{\in_\alpha} (\alpha \backslash \beta)$ uniquely exists by Corollary 3.4.

By Requirement 1.10, it now suffices to show the following two:

$\underline{\beta \subseteq \min_{\in_\alpha} (\alpha \backslash \beta)\text{:}}$ Suppose $\eta \in \beta$. As η, $\min_{\in_\alpha} (\alpha \backslash \beta) \in \alpha$, and $\in_\alpha$ is total on α by Lemma 3.3, to show $\eta \in \min_{\in_\alpha} (\alpha \backslash \beta)$, it suffices to refute both $\eta = \min_{\in_\alpha} (\alpha \backslash \beta)$ and $\min_{\in_\alpha} (\alpha \backslash \beta) \in \eta$. But in the first case trivially and in the second case as β is full, we would have $\min_{\in_\alpha} (\alpha \backslash \beta) \in \beta$, contradicting $\min_{\in_\alpha} (\alpha \backslash \beta) \in \alpha \backslash \beta$.

$\underline{\beta \supseteq \min_{\in_\alpha} (\alpha \backslash \beta)\text{:}}$ Suppose $\eta \in \min_{\in_\alpha} (\alpha \backslash \beta)$. As α is full, we have $\eta \in \alpha$ and $\eta \in_\alpha \min_{\in_\alpha} (\alpha \backslash \beta)$. Thus, $\eta \notin (\alpha \backslash \beta)$. Thus, $\eta \in \beta$.

Q.e.d. (Lemma 3.15)

Lemma 3.16 *If* $\alpha, \beta\ [\in \mathcal{S}\,]$ *are* $[\mathcal{S}$-$]$ Neumann *ordinals, then* $\alpha \subseteq \beta$ *or* $\beta \subseteq \alpha$.

Proof of Lemma 3.16
By Corollary 3.11, $\alpha \cap \beta$ is an $[\mathcal{S}$-$]$ Neumann ordinal as well. [Moreover, we have $\alpha \cap \beta \in \mathcal{S}$ and $\alpha \backslash (\alpha \cap \beta), \beta \backslash (\alpha \cap \beta) \in \mathcal{S}$ by the Axiom of Simple Operations (cf. Requirement 1.24).] As $\alpha \supseteq \alpha \cap \beta \subseteq \beta$, it suffices to refute $\alpha \supsetneq \alpha \cap \beta \subsetneq \beta$. But then, by Lemma 3.15, we would have $\alpha \ni \alpha \cap \beta \in \beta$, i.e. $\alpha \cap \beta \in \alpha \cap \beta$, contradicting Corollary 3.9. **Q.e.d. (Lemma 3.16)**

Lemma 3.17 *If* $\alpha, \beta\ [\in \mathcal{S}\,]$ *are* $[\mathcal{S}$-$]$ Neumann *ordinals, then exactly one of the following three cases holds:*

(i) $\alpha = \beta$.

(ii) $\alpha \in \beta$.

(iii) $\beta \in \alpha$.

Proof of Lemma 3.17
By Lemma 3.16, we have

$$\alpha = \beta \quad \vee \quad \alpha \subsetneq \beta \quad \vee \quad \beta \subsetneq \alpha.$$

[Moreover, we have $\alpha \cap \beta \in \mathcal{S}$ and $\alpha \backslash (\alpha \cap \beta), \beta \backslash (\alpha \cap \beta) \in \mathcal{S}$ by the Axiom of Simple Operations (cf. Requirement 1.24).] Thus, by Lemma 3.15, we have

$$\alpha = \beta \quad \vee \quad \alpha \in \beta \quad \vee \quad \beta \in \alpha.$$

If two of these cases fell together, we would get $\alpha \in \alpha$ or $\beta \in \beta$ (in case of the two latter cases, this is implied by fullness), contradicting Corollary 3.9. **Q.e.d. (Lemma 3.17)**

Lemma 3.18 α *is an* $\mathcal{S}$-Neumann *ordinal iff* α *is a* Neumann *ordinal, provided that we assume the Axiom of Separation.*

Proof of Lemma 3.18
The backward implication is given already by Corollary 3.8.
To show the forward implication, let us assume that α is an $\mathcal{S}$-Neumann ordinal and that $Q \subseteq \alpha$ with $\delta \in Q$.
All we have to find is some $\gamma \in Q$ such that $\forall \beta \in Q \backslash \{\gamma\}.\ (\gamma \in \beta)$. According to Lemmas 3.17 and 3.13, it now suffices to find some $\gamma \in Q$ such that $\forall \beta \in Q.\ (\beta \notin \gamma)$.
Set $Q' := \delta \cap Q$. Due to the Axiom of Separation (cf. Definition 1.28), we have $Q' \in \mathcal{S}$.
In case of $Q' = \emptyset$, for any $\beta \in Q$, we have $\beta \notin \delta$ as was to be shown.
Otherwise, in case of $Q' \neq \emptyset$, as $\in_\delta$ is $\mathcal{S}$-well-founded by Lemma 3.3(2), there is some $\gamma \in Q'$ with $\forall \beta \in Q'.\ (\beta \notin \gamma)$. Now let us assume $\beta \in Q$ and, *ad absurdum*, $\beta \in \gamma$. Then, due to $\gamma \in \delta$, as δ is full, we have $\beta \in \delta$. Thus, $\beta \in Q'$. Thus, $\beta \notin \gamma$.
Q.e.d. (Lemma 3.18)

The following proposition is quite trivial as a statement on [$\mathcal{S}$-] Neumann ordinals; but as the respective order-isomorphism cannot exist for huge [$\mathcal{S}$-] Frege ordinals in NF, it is interesting insofar as it implicitly says that all [$\mathcal{S}$-] Neumann ordinals are comparatively "small".

Lemma 3.19 *Let α be an [$\mathcal{S}$-] Neumann ordinal. Define* $\underline{\in}_\alpha := \in_\alpha \cup {}_\alpha\!\upharpoonleft\!\mathrm{id}$. *Define* $\subseteq_\alpha \; := \; \{ \; (B,C) \;\big|\; B \subseteq C \;\wedge\; \exists\beta\in\alpha.\, (B = \in_\alpha\langle\!\{\beta\}\!\rangle) \;\wedge\; \exists\gamma\in\alpha.\, (C = \in_\alpha\langle\!\{\gamma\}\!\rangle) \; \}$. *Now* $({}_\alpha\!\upharpoonleft\!\mathrm{id})\text{::}\underline{\in}_\alpha\!\to\subseteq_\alpha$ *is an order-isomorphism. Moreover, in case of* $\alpha\in\mathcal{S}$, *this is an* $\mathcal{S}$*-order-isomorphism, provided that we either assume the Set Comprehension Axiom of* ML*, or else the Axiom of Separation.*

Proof of Lemma 3.19
By Lemma 3.13, we have $\subseteq_\alpha \; = \{ \; (\beta,\gamma) \;|\; \beta\underline{\subseteq}\gamma \wedge \beta,\gamma\in\alpha \; \}$. Due to Lemma 3.13, we have $\forall\beta,\gamma\in\alpha.\ (\beta\underline{\in}_\alpha\gamma \Rightarrow \beta\underline{\subseteq}\gamma)$. Due to Lemma 3.15 and the Axiom of Simple Operations (cf. Requirement 1.24), we have $\forall\beta,\gamma\in\alpha.\ (\beta\underline{\in}_\alpha\gamma \Leftarrow \beta\underline{\subseteq}\gamma)$. Thus, $({}_\alpha\!\upharpoonleft\!\mathrm{id})\text{::}\underline{\in}_\alpha\!\to\subseteq_\alpha$ is an order-isomorphism. Finally, assume $\alpha\in\mathcal{S}$. By the Axiom of Simple Operations (cf. Requirement 1.24), we have $({}_\alpha\!\upharpoonleft\!\mathrm{id}),\ \alpha\times\alpha \in \mathcal{S}$. By Lemma 2.17, it now suffices to show $\subseteq_\alpha \in \mathcal{S}$. If we assume the Set Comprehension Axiom of ML, we get this from $\subseteq_\alpha \; = \{ \; p \;|\; p=(\beta,\gamma) \wedge \beta\underline{\subseteq}\gamma \wedge \beta,\gamma\in\alpha \; \}$. If we assume the Axiom of Separation, we get this from $\subseteq_\alpha \;\subseteq\; \alpha\times\alpha$. **Q.e.d. (Lemma 3.19)**

3.3 The Proper Class of Neumann Ordinals

Definition 3.20 ($\mathcal{O}$)
The *class of the* [$\mathcal{S}$-] Neumann *ordinals* is given as

$$[\mathcal{S}]\mathcal{O} \;:=\; \{ \; \alpha \;|\; \alpha \text{ is an } [\mathcal{S}\text{-}] \text{ Neumann ordinal } \}.$$

Corollary 3.21 $\mathcal{O} \subseteq \mathcal{S}\mathcal{O}$.

Corollary 3.22 $\mathcal{O} = \mathcal{S}\mathcal{O}$, *provided that we assume the Axiom of Separation.*

Theorem 3.23 $[\mathcal{S}]\mathcal{O}$ *is an* [$\mathcal{S}$-] Neumann *ordinal.*

Proof of Theorem 3.23
By Lemma 3.13, $[\mathcal{S}]\mathcal{O}$ is full.

It remains to show that $\in_{[\mathcal{S}]\mathcal{O}}$ is an irreflexive [$\mathcal{S}$-] well-ordering on $[\mathcal{S}]\mathcal{O}$. By Lemma 3.3(2), it suffices to show that $\in_{[\mathcal{S}]\mathcal{O}}$ is total on $[\mathcal{S}]\mathcal{O}$, transitive, and [$\mathcal{S}$-] well-founded.

For transitivity, suppose $\gamma \in_{[\mathcal{S}]\mathcal{O}} \beta \in_{[\mathcal{S}]\mathcal{O}} \alpha$. Then $[\mathcal{S}]\mathcal{O} \ni \gamma \in \beta \in \alpha \in [\mathcal{S}]\mathcal{O}$. Thus, as α is full, we have $\gamma \in_{[\mathcal{S}]\mathcal{O}} \alpha$.

The required totality is given by Lemma 3.17.

Finally, for $[\mathcal{S}$-$]$ well-foundedness, suppose $Q \subseteq [\mathcal{S}]\mathcal{O}$ with $\delta \in Q$ [and $Q \in \mathcal{S}$].
All we have to find is some $\gamma \in Q$ such that $\forall\beta \in Q\backslash\{\gamma\}.\ (\gamma \in \beta)$. According to Lemmas 3.17 and 3.13, it now suffices to find some $\gamma \in Q$ such that $\forall\beta \in Q.\ (\beta \notin \gamma)$.
Set $Q' := \delta \cap Q$.
[Due to the Axiom of Simple Operations (cf. Requirement 1.24) we have $Q' \in \mathcal{S}$.]
In case of $Q' = \emptyset$, for any $\beta \in Q$, we have $\beta \notin \delta$ as was to be shown.
Otherwise, in case of $Q' \neq \emptyset$, as $\in_\delta$ is $[\mathcal{S}$-$]$ well-founded by Lemma 3.3(2), there is some $\gamma \in Q'$ with $\forall\beta \in Q'.\ (\beta \notin \gamma)$. Now let us assume $\beta \in Q$ and, *ad absurdum*, $\beta \in \gamma$. Then, due to $\gamma \in \delta$, as δ is full, we have $\beta \in \delta$. Thus, $\beta \in Q'$. Thus, $\beta \notin \gamma$.
Q.e.d. (Theorem 3.23)

As a corollary of Corollary 3.9, we get:

Corollary 3.24 $[\mathcal{S}]\mathcal{O} \notin [\mathcal{S}]\mathcal{O}$.

Corollary 3.25 *$[\mathcal{S}]\mathcal{O}$ is a proper class, i.e. $[\mathcal{S}]\mathcal{O} \notin \mathcal{V}$.*

Corollary 3.26 *$[\mathcal{S}]\mathcal{O} \cup \{[\mathcal{S}]\mathcal{O}\} = [\mathcal{S}]\mathcal{O}$ and there is no Ω with $[\mathcal{S}]\mathcal{O} \in \Omega$.*

Definition 3.27 (Limit Ordinal)
α is a *limit* $[\mathcal{S}$-$]$ NEUMANN *ordinal* if
α is an $[\mathcal{S}$-$]$ NEUMANN ordinal and $\neg\exists\beta \in [\mathcal{S}]\mathcal{O}.\ (\alpha = \beta \cup \{\beta\})$.

Note that we treat $\emptyset$ as a limit ordinal, which is not quite standard, but most reasonable. Moreover, note that, in Definition 3.27, it is important to require $\beta \in \mathcal{V}$ besides that β is an $[\mathcal{S}$-$]$ NEUMANN ordinal, because otherwise any $[\mathcal{S}$-$]$ NEUMANN ordinal $\alpha \notin \mathcal{V}$ would *not* be limit ordinal, due to $\alpha = \alpha \cup \{\alpha\}$.

Lemma 3.28 *$[\mathcal{S}]\mathcal{O}$ is a limit $[\mathcal{S}$-$]$ NEUMANN ordinal.*

Proof of Lemma 3.28
Otherwise, due to Theorem 3.23, there would be a $\beta \in [\mathcal{S}]\mathcal{O}$ with $\beta \cup \{\beta\} = [\mathcal{S}]\mathcal{O}$. But then we have $\beta \in \mathcal{V}$ due to $\beta \in [\mathcal{S}]\mathcal{O}$, moreover $\{\beta\} \in \mathcal{V}$ due to Requirement 1.21, and then $\beta \cup \{\beta\} \in \mathcal{V}$ due to Requirement 1.24. But then $[\mathcal{S}]\mathcal{O} \in \mathcal{V}$, contradicting Corollary 3.25. **Q.e.d. (Lemma 3.28)**

3.4 Basic Properties of Well-Orderings

Initial segments were introduced already in Definition 2.18, but now they are going to become essential.

Corollary 3.29 *An initial segment of an $[\mathcal{S}$-$]$ well-ordering on A is an $[\mathcal{S}$-$]$ well-ordering on a proper subclass of A.*

Lemma 3.30 *Let $\leq$ be an $[\mathcal{S}$-$]$ well-ordering, and let $<$ be its ordering.*
If $a{<}b$, then $\lfloor\leq\rfloor_a$ is the initial segment of $\lfloor\leq\rfloor_b$ below a.

Although this lemma seems to be trivial, note that it would not hold if we had defined initial segments via irreflexive well-orderings: For the irreflexive well-ordering $< := \{(0,1)\}$, we would have $\lfloor<\rfloor_0 = \emptyset = \lfloor<\rfloor_1$, so that $\lfloor<\rfloor_0$ would not be an initial segment of $\lfloor<\rfloor_1$.

Proof of Lemma 3.30
From $a{<}b$, we get $a \in B$ for $B := {<}\langle\!\langle\{b\}\rangle\!\rangle$. Thus — and this is the non-trivial step! — $a \in \mathrm{field}(\lfloor\leq\rfloor_b)$. Thus, $\lfloor\leq\rfloor_a$ is the initial segment of $\lfloor\leq\rfloor_b$ below a. **Q.e.d. (Lemma 3.30)**

Lemma 3.31
No $[\mathcal{S}$-$]$ well-ordering is $[\mathcal{S}$-$]$ order-isomorphic to an initial segment of itself $[$provided that we assume either the Set Comprehension Axiom of ML*, or else the Axiom of Separation$]$.*

Proof of Lemma 3.31 Let $\leq$ be an $[\mathcal{S}$-$]$ well-ordering, and let $<$ be its ordering. Then $<$ is $[\mathcal{S}$-$]$ well-founded according to Lemma 3.3. For a *reductio ad absurdum*, suppose $f{::}{\leq}{\rightarrow}\lfloor\leq\rfloor_a$ to be an $[\mathcal{S}$-$]$ order-isomorphism for some $a \in \mathrm{field}(\leq)$. Then $f(a){<}a$. By Corollary 2.16, we have $\forall a, b.\ (a{<}b \ \Rightarrow\ f(a) < f(b))$. By Lemma 2.5, we get the contradictory $\forall x.\ \neg(f(x){<}x)$ [provided that we either have the Set Comprehension Axiom of ML and $<, f \in \mathcal{S}$, or else the Axiom of Separation and $\mathrm{field}(<) \in \mathcal{S}$]. [Note that we have $f, {\leq} \in \mathcal{S}$. We get $<$, $\mathrm{field}(<) \in \mathcal{S}$ from $< \ = \ \leq\backslash({}_{\mathrm{field}(\leq)}\!\upharpoonright\!\mathrm{id})$ by Requirement 1.24 and Corollaries 1.25 and 2.10.] **Q.e.d. (Lemma 3.31)**

Lemma 3.32 *Let $\leq$ be an $[\mathcal{S}$-$]$ well-ordering. Let $a, b \in \mathrm{field}(\leq)$.*
[Assume either the Set Comprehension Axiom of ML*, or else the Axiom of Separation.]*

(1) If $\lfloor\leq\rfloor_a$ and $\lfloor\leq\rfloor_b$ are $[\mathcal{S}$-$]$ order-isomorphic, then $a{=}b$.

(2) If $\lfloor\leq\rfloor_a$ is $[\mathcal{S}$-$]$ order-isomorphic to an initial segment of $\lfloor\leq\rfloor_b$, then $a{<}b$.

Proof of Lemma 3.32
Let $<$ be be the ordering of $\leq$. By Lemma 3.3, $\leq$ and $<$ are total on $\mathrm{field}(\leq)$.

(1): Suppose that $\lfloor\leq\rfloor_a$ and $\lfloor\leq\rfloor_b$ are $[\mathcal{S}$-$]$ order-isomorphic. As both $a{<}b$ and $a{>}b$ contradict Lemmas 3.30 and 3.31, we have $a{=}b$.

(2): Suppose that $\lfloor\leq\rfloor_a$ and $\leq'$ are $[\mathcal{S}$-$]$ order-isomorphic, and that $\leq'$ is an initial segment of $\lfloor\leq\rfloor_b$. Then there is some $c < b$ such that $\leq' \ = \lfloor\leq\rfloor_c$. Then $a{=}c$ by (1). **Q.e.d. (Lemma 3.32)**

Theorem 3.33
[*Assume either the Set Comprehension Axiom of* ML, *or else the Axiom of Separation.*]
If $\leq_0, \leq_1$ $[\in\mathcal{S}]$ *are two* $[\mathcal{S}$-$]$ *well-orderings, then exactly one of the following three cases holds:*

(i) $\leq_0, \leq_1$ *are* $[\mathcal{S}$-$]$ *order-isomorphic.*

(ii) $\leq_0$ *is* $[\mathcal{S}$-$]$ *order-isomorphic to an initial segment of* $\leq_1$*.*

(iii) $\leq_1$ *is* $[\mathcal{S}$-$]$ *order-isomorphic to an initial segment of* $\leq_0$*.*

Proof of Theorem 3.33
Let $<_i$ be the ordering of the well-ordering $\leq_i$ $(i\in\{0,1\})$. [By the Corollary 1.25 and Requirement 1.24, $\mathrm{field}(\leq_0)$, $\mathrm{field}(\leq_1)$, $\mathrm{field}(\leq_0)\times\mathrm{field}(\leq_1)\in\mathcal{S}$.] We set

$$f \;:=\; \left\{\;(a_0,a_1)\in\mathrm{field}(\leq_0)\times\mathrm{field}(\leq_1)\;\middle|\;\lfloor\leq_0\rfloor_{a_0}\text{ is }[\mathcal{S}\text{-}]\text{ order-isomorphic to }\lfloor\leq_1\rfloor_{a_1}\;\right\}$$

[We have $f\in\mathcal{S}$, either due to the Set Comprehension Axiom of ML or else due to the Axiom of Separation. By Requirement 1.24 and Corollary 1.25, $\mathrm{dom}(f)$, $\mathrm{ran}(f)\in\mathcal{S}$, and then $\mathrm{field}(\leq_0)\setminus\mathrm{dom}(f)$, $\mathrm{field}(\leq_1)\setminus\mathrm{ran}(f)\in\mathcal{S}$.]

Note that by Corollary 2.15, we have symmetry in f and f^{-1}.

<u>Claim 1:</u> f is an injective function.

<u>Proof of Claim 1:</u> By symmetry in f and f^{-1}, it suffices to show that f is a function. To show this, suppose that $(a_0,a_1),(a_0,b_1)\in f$. By Corollary 2.15, $\lfloor\leq_1\rfloor_{a_1}$ and $\lfloor\leq_1\rfloor_{b_1}$ are $[\mathcal{S}$-$]$ order-isomorphic. By Lemma 3.32, we have $a_1=b_1$. <u>Q.e.d. (Claim 1)</u>

<u>Claim 2:</u> $\forall b\in\mathrm{dom}(f).\ \forall a<_0 b.\ \big(\ a\in\mathrm{dom}(f)\ \ \wedge\ \ f(a)<_1 f(b)\ \big)$ and
$\forall b\in\mathrm{ran}(f).\ \forall a<_1 b.\ \big(\ a\in\mathrm{ran}(f)\ \ \wedge\ \ f^{-1}(a)<_0 f^{-1}(b)\ \big)$.

<u>Proof of Claim 2:</u> By symmetry in f and f^{-1}, it suffices to show the first statement. Let $b\in\mathrm{dom}(f)$. Then there is some order-isomorphism $h{::}\lfloor\leq_0\rfloor_b\to\lfloor\leq_1\rfloor_{f(b)}$. Suppose $a<_0 b$. By Lemma 2.20 we have $\langle<_0\langle\!\{a\}\!\rangle\rangle h = {<_1}\langle\!\{h(a)\}\!\rangle$. Thus, $({}_{<_0\langle\!\{a\}\!\rangle}{\restriction}h){::}\lfloor\leq_0\rfloor_a\to\lfloor\leq_1\rfloor_{h(a)}$ is an order-isomorphism. Thus, $f(a)=h(a)<_1 f(b)$. <u>Q.e.d. (Claim 2)</u>

<u>Claim 3:</u> We have $\mathrm{dom}(f)=\mathrm{field}(\leq_0)$ or $\mathrm{ran}(f)=\mathrm{field}(\leq_1)$.
If $\mathrm{dom}(f)=\mathrm{field}(\leq_0)$ and $\mathrm{ran}(f)=\mathrm{field}(\leq_1)$, then $f{::}\leq_0\to\leq_1$ is an $[\mathcal{S}$-$]$ order-isomorphism. If $\mathrm{ran}(f)\neq\mathrm{field}(\leq_1)$, then there is some $a_1\in\mathrm{field}(\leq_1)$ such that $f{::}\leq_0\to\lfloor\leq_1\rfloor_{a_1}$ is an $[\mathcal{S}$-$]$ order-isomorphism. If $\mathrm{dom}(f)\neq\mathrm{field}(\leq_0)$, then there is some $a_0\in\mathrm{field}(\leq_0)$ such that $f{::}\lfloor\leq_0\rfloor_{a_0}\to\leq_1$ is an $[\mathcal{S}$-$]$ order-isomorphism.

<u>Proof of Claim 3:</u> Now it becomes crucial that $\leq_0,\leq_0$ are $[\mathcal{S}$-$]$ well-orderings. Be reminded of Corollary 3.4. In case of $\mathrm{dom}(f)\neq\mathrm{field}(\leq_0)$, set $a_0:=\min_{\leq_0}(\mathrm{field}(\leq_0)\setminus\mathrm{dom}(f))$. In case of $\mathrm{ran}(f)\neq\mathrm{field}(\leq_1)$, set $a_1:=\min_{\leq_1}(\mathrm{field}(\leq_1)\setminus\mathrm{ran}(f))$. Then, by Claim 2, $\mathrm{dom}(f)={<_0}\langle\!\{a_0\}\!\rangle$, $\mathrm{ran}(f)={<_1}\langle\!\{a_1\}\!\rangle$, respectively. In the case of both, by Claims 1 and 2, $f{::}\lfloor\leq_0\rfloor_{a_0}\to\lfloor\leq_1\rfloor_{a_1}$ would be an $[\mathcal{S}$-$]$ order-isomorphism, and then we would have the contradictory $a_0\in\mathrm{dom}(f)$. Thus, we get the given cases and the described order-isomorphisms by Claims 1 and 2 [and Lemma 2.17]. <u>Q.e.d. (Claim 3)</u>

By Claim 3, all that is left to show is that the three case are mutually disjoint. But case (i) trivially does not go together with (ii) or (iii) by Corollary 2.15 and Lemma 3.31. Similarly, if cases (ii) and (iii) fell together, say $h_i{::}\leq_i\to\lfloor\leq_{1-i}\rfloor_{a_{1-i}}$ were $[\mathcal{S}$-$]$ order-isomorphisms for $i\in\{0,1\}$, then $(h_0\circ h_1){::}\leq_0\to\lfloor\leq_0\rfloor_{h_1(a_1)}$ would be an $[\mathcal{S}$-$]$ order-isomorphism by Lemmas 2.21 and 3.30, again contradicting Lemma 3.31. **Q.e.d. (Theorem 3.33)**

3.5 Frege **Ordinals**

It is a pity that in his seminal book [Quine, 1981] on ML, Willard Van O. Quine (1908–2000) does not treat any infinite ordinal or cardinal numbers. In his finite cardinals, however, Quine follows Gottlob Frege (1848–1925) [Frege, 1884; 1893/1903]; and we will call cardinal and ordinal numbers in this style "Frege cardinals" and "Frege ordinals", respectively.

A Frege ordinal is the most natural idea of an ordinal number of a given well-ordering $\leq$, namely the class of all reflexive orderings that are order-isomorphic to $\leq$, i.e. the order-type $\mathcal{OT}(\leq, X)$ of $\leq$ over some class X; cf. Definition 2.22.

Definition 3.34 (Frege **Ordinals over** X, **"**$\mathcal{FO}(X)$**"**)
α is an $[\mathcal{S}$-$]$ Frege *ordinal over* X if

$$\neg\,\mathcal{U}(X) \text{ and there is an } [\mathcal{S}\text{-}] \text{ well-ordering } \leq\,\in\alpha \text{ such that } \alpha = \mathcal{OT}(\leq, X).$$

$[\mathcal{S}]\mathcal{FO}(X) := \{\ \alpha \mid \alpha \text{ is an } [\mathcal{S}\text{-}] \text{ Frege ordinal over } X\ \}$.

Corollary 3.35

(1) If α *is a* Frege *ordinal over* X*,*
then $\neg\,\mathcal{U}(\alpha)$ *and* α *is an* $\mathcal{S}$-Frege *ordinal over* X *as well.*

(2) $\mathcal{FO}(X) \subseteq \mathcal{SFO}(X) \subseteq \mathcal{S}$.

Corollary 3.36 *Let* $X \in \mathcal{S}$. *Assume the Axiom of Separation. Then:*

(1) α *is a* Frege *ordinal over* X *iff* α *is an* $\mathcal{S}$-Frege *ordinal over* X.

(2) $\mathcal{FO}(X) = \mathcal{SFO}(X)$.

In Definition 3.34, we have restricted the fields of the well-orderings to subclasses of a class X for two reasons: The first is that we need $\mathcal{SFO}(X)$ for a set X in Hartogs' Ordinal Theorem, cf. Theorem 3.46. The second reason is that in the NBG and MK class theories, $[\mathcal{S}$-$]$ Frege ordinals over a proper class, such as $\mathcal{V}$, are not much fun because they are all proper classes, with the exception of the Frege ordinal $\{\emptyset\}$ of the empty ordering $\emptyset$. This means that in NBG and MK we have $\mathcal{FO}(\mathcal{V}) = \mathcal{SFO}(\mathcal{V}) = \{\{\emptyset\}\}$. For a set X, however, as a corollary of Lemma 2.24 we get:

Corollary 3.37 *If* α *is an* $[\mathcal{S}$-$]$ Frege *ordinal over* $X \in \mathcal{S}$, *then* $\alpha \in [\mathcal{S}]\mathcal{FO}(X)$, *provided that we assume either the Set Comprehension Axiom of* ML, *or else both the Power-Set Axiom and the Axiom of Separation.*

Definition 3.38 (Ordering of $\mathcal{S}$-Frege **Ordinals, "**$\prec_{\mathcal{SFO}(X)}$**")**

$$\prec_{[\mathcal{S}]\mathcal{FO}(X)} := \left\{ (\alpha_0, \alpha_1) \;\middle|\; \begin{array}{l} \alpha_0, \alpha_1 \in [\mathcal{S}]\mathcal{FO}(X) \\ \wedge\ \exists{\leq_0} \in \alpha_0.\ \exists{\leq_1} \in \alpha_1.\ \exists a_1 \in \mathrm{field}(\leq_1). \\ \qquad (\leq_0 \text{ is } [\mathcal{S}\text{-}] \text{ order-isomorphic to } \lfloor\leq_1\rfloor_{a_1}) \end{array} \right\}.$$

$\preceq_{\mathcal{SFO}(X)} := \prec_{\mathcal{SFO}(X)} \cup {}_{\mathcal{SFO}(X)}{\upharpoonleft}\mathrm{id}$.

Corollary 3.39 *Let us assume the Axiom of Separation.*
Then $\prec_{\mathcal{FO}(X)} = {}_{\mathcal{FO}(X)}{\upharpoonleft}{\prec_{\mathcal{SFO}(X)}}{\upharpoonright}_{\mathcal{FO}(X)}$.
Moreover, if $X \in \mathcal{S}$, *then* $\prec_{\mathcal{FO}(X)} = \prec_{\mathcal{SFO}(X)}$.

Remark 3.40 Note that we have not defined a reflexive ordering $\preceq_{\mathcal{FO}(X)}$ in Definition 3.38 because this would not have any interesting properties:

1. According to Definitions 3.34 and 2.22, to show $\lfloor\leq_1\rfloor_{a_1} \in \alpha_0$ in case of $\leq_0 \in \alpha_0$, it is not sufficient that $\leq_0$ is order-isomorphic to $\lfloor\leq_1\rfloor_{a_1}$. Instead we need that $\leq_0$ is $\mathcal{S}$-order-isomorphic to $\lfloor\leq_1\rfloor_{a_1}$. And this would cause similar problems, most easily to be overlooked. For instance, already Claim 1 of the Proof of Lemma 3.45 would not hold. As the removal of the "$\mathcal{S}$-" in Definition 2.22 would deprive us of the elementship of order-types of Lemma 2.24, and thereby render order-types practically useless, the only way to proceed here would be to define a quasi-ordering $\lesssim_{\mathcal{FO}(X)}$, where two order-types are equivalent iff they are order-isomorphic. The theory of $\lesssim_{\mathcal{FO}(X)}$ would be awkward, however, even after extending our theory of well-orderings to such quasi-well-orderings. Note that $\prec_{\mathcal{FO}(X)}$ is not necessarily an irreflexive well-ordering because it has no minimum in non-trivial equivalence classes of $\lesssim_{\mathcal{FO}(X)}$.
2. If we assume the Axiom of Separation, by Corollary 3.39, then there is hardly any reason to consider $\prec_{\mathcal{FO}(X)}$ in addition to $\prec_{\mathcal{SFO}(X)}$.
3. If we do not assume the Axiom of Separation, however, the optional case of Definition 3.38 is quite unimportant anyway, because we cannot show $\mathcal{FO}(X) \in \mathcal{S}$ for $X \in \mathcal{S}$.

Lemma 3.41
[*Assume either the Set Comprehension Axiom of* ML*, or else the Axiom of Separation.*]
$\prec_{[\mathcal{S}]\mathcal{FO}(X)}$ *is an ordering on* $[\mathcal{S}]\mathcal{FO}(X)$.

Proof of Lemma 3.41 By Corollary 2.15 and Lemma 3.31, $\prec_{[\mathcal{S}]\mathcal{FO}(X)}$ is irreflexive. By Corollary 2.15 and Lemma 2.21, $\prec_{[\mathcal{S}]\mathcal{FO}(X)}$ is transitive. **Q.e.d. (Lemma 3.41)**

Lemma 3.42
Assume either the Set Comprehension Axiom of ML*, or else the Axiom of Separation.*
$\preceq_{\mathcal{SFO}(X)}$ *is an* $\mathcal{S}$*-well-ordering on* $\mathcal{SFO}(X)$.
$\prec_{\mathcal{SFO}(X)}$ *is an irreflexive* $\mathcal{S}$*-well-ordering on* $\mathcal{SFO}(X)$*, and the ordering of* $\preceq_{\mathcal{SFO}(X)}$.

Proof of Lemma 3.42

By Lemma 3.41, $\prec_{\mathcal{SFO}(X)}$ is an ordering on $\mathcal{SFO}(X)$. Thus, by Corollary 2.11, $\preceq_{\mathcal{SFO}(X)}$ is a reflexive ordering on $\mathcal{SFO}(X)$. By Corollary 2.10, $\prec_{\mathcal{SFO}(X)}$ is the ordering of $\preceq_{\mathcal{SFO}(X)}$. By Lemma 3.3(4), it now suffices to show that $\preceq_{\mathcal{SFO}(X)}$ is an $\mathcal{S}$-well-ordering on $\mathcal{SFO}(X)$.

To this end, let us assume $\alpha_0 \in Q \subseteq \mathcal{SFO}(X)$ and $Q \in \mathcal{S}$. Then α_0 is an $\mathcal{S}$-Frege ordinal over the class X. Then there is an $\mathcal{S}$-well-ordering $\leq_0 \in \alpha_0$. Set

$$Q' := \{\ a \in \mathrm{field}(\leq_0) \mid \exists\alpha_1 \in Q.\ \exists{\leq_1} \in \alpha_1.\ (\leq_1 \text{ is } \mathcal{S}\text{-order-isomorphic to } \lfloor\leq_0\rfloor_a)\ \}.$$

Due to $\leq_0 \in \mathcal{S}$, by the Axiom of Simple Operations and either the Set Comprehension Axiom of ML or else the Axiom of Separation, we get $\mathrm{field}(\leq_0), Q' \in \mathcal{S}$.

In case of $Q' = \emptyset$, we set $\alpha'_0 := \alpha_0$ and $\leq'_0 := \leq_0$.

In case of $Q' \neq \emptyset$, however, we set $m := \min_{\leq_0} Q'$ by Corollary 3.4. Then, due to $m \in Q'$, there are $\alpha'_0 \in Q$, $\leq'_0 \in \alpha'_0$, and an $\mathcal{S}$-order-isomorphism $f_1 :: \leq'_0 \to \lfloor\leq_0\rfloor_m$.

In any case, let $\alpha_1 \in Q$ be arbitrary. As $\alpha_1 \in Q$, there is an $\mathcal{S}$-well-ordering $\leq_1 \in \alpha_1$. It suffices to show $\alpha'_0 \preceq_{\mathcal{SFO}(X)} \alpha_1$. By Theorem 3.33, we have three cases now:

If $\leq'_0$ is $\mathcal{S}$-order-isomorphic to $\leq_1$, then, due to $\leq'_0 \in \alpha'_0$, $\leq_1 \in \alpha_1$, Definitions 3.34 and 2.22, and Corollary 2.15, we get $\alpha'_0 = \alpha_1$, which implies $\alpha'_0 \preceq_{\mathcal{SFO}(X)} \alpha_1$.

If $\leq'_0$ is $\mathcal{S}$-order-isomorphic to an initial segment of $\leq_1$, then $\alpha'_0 \prec_{\mathcal{SFO}(X)} \alpha_1$, which implies $\alpha'_0 \preceq_{\mathcal{SFO}(X)} \alpha_1$.

Otherwise, there is an $\mathcal{S}$-order-isomorphism $f_0{::}{\leq_1}{\rightarrow}\lfloor\leq'_0\rfloor_b$ for some $b \in \mathrm{field}(\leq'_0)$. In case of $Q' = \emptyset$, we have $\leq'_0 = \leq_0$, and then the contradictory $b \in Q'$. Thus, $Q' \neq \emptyset$. Then, by Lemma 2.21, $(f_0{\circ}f_1){::}{\leq_1}{\rightarrow}\lfloor\leq_0\rfloor_{f_1(b)}$ is an $\mathcal{S}$-order-isomorphism with $f_1(b) <_0 m$. But then $f_1(b) \in Q'$, which contradicts the description of m. **Q.e.d. (Lemma 3.42)**

Lemma 3.43 *If* $X \in \mathcal{S}$, *then* $\mathcal{SFO}(X)$, $\prec_{\mathcal{SFO}(X)}$, $\preceq_{\mathcal{SFO}(X)}$ $\in \mathcal{S}$, *provided that we assume either the Set Comprehension Axiom of* ML, *or else both the Power-Set Axiom and the Axiom of Separation.*

Proof of Lemma 3.43

By the Axiom of Simple Operations, the Power-Set Axiom, and the Axiom of Separation, we get $[\mathcal{S}]\mathcal{FO}(X) \in \mathcal{S}$ from $\mathcal{FO}(X) \subseteq \mathcal{SFO}(X) \subseteq \mathfrak{P}(\mathfrak{P}(X{\times}X)) \in \mathcal{S}$; and then $\prec_{[\mathcal{S}]\mathcal{FO}(X)}$, $\preceq_{\mathcal{SFO}(X)} \in \mathcal{S}$ from $\prec_{[\mathcal{S}]\mathcal{FO}(X)}$, $\preceq_{\mathcal{SFO}(X)} \subseteq \mathcal{SFO}(X) \times \mathcal{SFO}(X) \in \mathcal{S}$.

For $\mathcal{SFO}(X) \in \mathcal{S}$, by the Set Comprehension Axiom of ML, we just have to show that the defining statement of Definition 3.34 is stratified and equivalent to the version where all quantifiers are restricted to elementship. Stratification is easy: Just set $\leq, X$ to the same integer number n; and α to $n{+}1$. The rest is follows from the Proof of Lemma 2.24.

Moreover, $\prec_{\mathcal{SFO}(X)} \in \mathcal{S}$ now easily follows from the Set Comprehension Axiom of ML and the Axiom of the Ordered Pair.

Finally, $\preceq_{\mathcal{SFO}(X)} \in \mathcal{S}$ now easily follows from the Axiom of Simple Operations.

Q.e.d. (Lemma 3.43)

3.6 Hartogs' Ordinal Theorem

The following Lemma 3.45 and Theorem 3.46 are due to Friedrich Hartogs (1874–1943).

Definition 3.44 (Hartogs-Morphism)
m is the X-Hartogs-*morphism onto* $\leq$ if $m = (f{::}\lfloor\preceq_{\mathcal{SFO}(X)}\rfloor_\alpha{\rightarrow}\leq)$ for

$$\begin{array}{rcl} f & := & \left\{ (\beta,b) \left| \begin{array}{ll} & \beta \prec_{\mathcal{SFO}(X)} \alpha \\ \wedge & b \in \mathrm{field}(\leq) \\ \wedge & \exists{\leq'}\in\beta.\ ({\leq'} \text{ is } \mathcal{S}\text{-order-isomorphic to } \lfloor\leq\rfloor_b) \end{array} \right. \right\}, \\ \alpha & := & \mathcal{OT}(\leq, X). \end{array}$$

Note that the Set Comprehension Axiom of ML does not seem to be sufficient to show $f\in\mathcal{S}$ for the f of Definition 3.44, because stratification of the defining formula with a substitution σ results in $\beta\sigma = \alpha\sigma = {\leq}\sigma + 1 = b\sigma + 2$, which is inconsistent with the requirement of $\beta\sigma = b\sigma$, resulting from the pair (β,b).

Lemma 3.45 ([Hartogs, 1915])
Let us assume either the Set Comprehension Axiom of ML, *or else both the Power-Set Axiom and the Axiom of Separation. Moreover, let us assume* $X\in\mathcal{S}$. *Then:*

(1) For every $\alpha\in\mathcal{SFO}(X)$ *and every* ${\leq}\in\alpha$,
the X-Hartogs-*morphism onto* $\leq$ *is an order-isomorphism.*

(2) If there is

$$\text{an injective function } \pi : \mathcal{SFO}(X) \rightarrow X \text{ with } \pi\in\mathcal{S},$$

then for

$$\begin{array}{rcl} \leq & := & \pi^{-1}\circ\preceq_{\mathcal{SFO}(X)}\circ\,\pi, \\ \alpha & := & \mathcal{OT}(\leq, X), \end{array}$$

we have

$${\leq} \in \alpha \in \mathcal{SFO}(X),$$

and

$$\pi{::}\preceq_{\mathcal{SFO}(X)}{\rightarrow}\leq \text{ is an } \mathcal{S}\text{-order-isomorphism,}$$

but there is

$$\text{no } \mathcal{S}\text{-order-isomorphism } f{::}\lfloor\preceq_{\mathcal{SFO}(X)}\rfloor_\alpha{\rightarrow}\leq.$$

Proof of Lemma 3.45

(1): Let ${\leq}\in\alpha\in\mathcal{SFO}(X)$. Then $\leq$ is an $\mathcal{S}$-well-ordering on a subclass of X.
Let $f{::}\lfloor\preceq_{\mathcal{SFO}(X)}\rfloor_\alpha{\rightarrow}\leq$ be the X-Hartogs-morphism onto $\leq$.

Claim 1: $\mathrm{dom}(f) = \mathrm{field}(\lfloor\preceq_{\mathcal{SFO}(X)}\rfloor_\alpha)$.

Proof of Claim 1: Let $\beta \prec_{\mathcal{SFO}(X)} \alpha$ be arbitrary. By Lemma 3.42, it suffices to show $\beta\in\mathrm{dom}(f)$. By Definition 3.38, there are ${\leq'}\in\beta$, ${\leq''}\in\alpha$, $b''\in\mathrm{field}(\leq'')$, and an $\mathcal{S}$-order-isomorphism $g_0{::}{\leq'}{\rightarrow}\lfloor\leq''\rfloor_{b''}$. Due to ${\leq},{\leq''}\in\alpha\in\mathcal{SFO}(X)$, by Definitions 3.34 and 2.22 and Corollary 2.15, there is an $\mathcal{S}$-order-isomorphism $g_1{::}{\leq''}{\rightarrow}\leq$. By Lemma 2.21, $(g_0{\circ}g_1){::}{\leq'}{\rightarrow}\lfloor\leq\rfloor_{g_1(b'')}$ is an $\mathcal{S}$-order-isomorphism and $g_1(b'')\in\mathrm{field}(\leq)$. Thus, $(\beta, g_1(b''))\in f$. Thus, $\beta\in\mathrm{dom}(f)$, as was to be shown. Q.e.d. (Claim 1)

Claim 2: f is a function.

Proof of Claim 2: In case that $(\beta, b_i) \in f$, for each $i \in \{1,2\}$, there are $\leq'_i \in \beta \in \mathcal{SFO}(X)$ such that $\leq'_i$ is $\mathcal{S}$-order-isomorphic to $\lfloor\leq\rfloor_{b_i}$. By Definitions 3.34 and 2.22 and Corollary 2.15, $\lfloor\leq\rfloor_{b_1}$ is $\mathcal{S}$-order-isomorphic to $\lfloor\leq\rfloor_{b_2}$. Thus, by Lemma 3.32(a), we have $b_1 = b_2$, as was to be shown. Q.e.d. (Claim 2)

Claim 3: f is injective.

Proof of Claim 3: In case that $(\beta_i, b) \in f$, for each $i \in \{1,2\}$, there are $\leq'_i \in \beta_i \in \mathcal{SFO}(X)$ such that $\leq'_i$ is $\mathcal{S}$-order-isomorphic to $\lfloor\leq\rfloor_b$. Then, by Corollary 2.15, $\leq'_1$ is $\mathcal{S}$-order-isomorphic to $\leq'_2$. Thus, by Definitions 3.34 and 2.22 and Corollary 2.15, we have $\beta_1 = \beta_2$, as was to be shown. Q.e.d. (Claim 3)

Claim 4: $\mathrm{ran}(f) = \mathrm{field}(\leq)$.

Proof of Claim 4: Let $b \in \mathrm{field}(\leq)$ be arbitrary. Set $\beta := \mathcal{OT}(\lfloor\leq\rfloor_b, X)$. By Corollary 3.29, $\lfloor\leq\rfloor_b$ is an $\mathcal{S}$-well-ordering on a proper subclass of X. By Lemma 2.19, we have $\lfloor\leq\rfloor_b \in \mathcal{S}$, and thus $\lfloor\leq\rfloor_b \in \beta$ by Corollary 2.23. Due to $X \in \mathcal{S}$, by Corollary 3.37, we have $\beta \in \mathcal{SFO}(X)$. By Corollary 2.15, $\lfloor\leq\rfloor_b$ is $\mathcal{S}$-order-isomorphic to $\lfloor\leq\rfloor_b$. Thus, by Definition 3.38, $\beta \prec_{\mathcal{SFO}(X)} \alpha$. All in all, we have $(\beta, b) \in f$. Thus, $b \in \mathrm{ran}(f)$, as was to be shown. Q.e.d. (Claim 4)

Let us assume γ, β to be arbitrary with $\gamma \prec_{\mathcal{SFO}(X)} \beta \prec_{\mathcal{SFO}(X)} \alpha$. For $f{::}\lfloor\preceq_{\mathcal{SFO}(X)}\rfloor_\alpha \to \leq$ to be an order-isomorphism, by Corollary 2.16, Claims 1–4, and Lemma 3.3(1b), it suffices to show $f(\gamma) < f(\beta)$. Due to $\gamma \prec_{\mathcal{SFO}(X)} \beta$, by Definition 3.38, there are $\leq'_\gamma \in \gamma$, $\leq'_\beta \in \beta$, $b \in \mathrm{field}(\leq'_\beta)$, and an $\mathcal{S}$-order-isomorphism $g_2{::}\leq'_\gamma \to \lfloor\leq'_\beta\rfloor_b$. Moreover, by definition of f and Corollary 2.15, there are $\leq''_\gamma \in \gamma$, $\leq''_\beta \in \beta$, and $\mathcal{S}$-order-isomorphisms $g_4{::}\leq''_\beta \to \lfloor\leq\rfloor_{f(\beta)}$ and $g_0{::}\lfloor\leq\rfloor_{f(\gamma)} \to \leq''_\gamma$. Furthermore, due to $\gamma, \beta \in \mathcal{SFO}(X)$, by Definitions 3.34 and 2.22 and Corollary 2.15, there are $\mathcal{S}$-order-isomorphisms $g_1{::}\leq''_\gamma \to \leq'_\gamma$ and $g_3{::}\leq'_\beta \to \leq''_\beta$. Then we get the $\mathcal{S}$-order-isomorphism $(g_3 \circ g_4){::}\leq'_\beta \to \lfloor\leq\rfloor_{f(\beta)}$. Then, by Lemma 2.21, $(g_2 \circ g_3 \circ g_4){::}\leq'_\gamma \to \lfloor\leq\rfloor_{(g_3 \circ g_4)(b)}$ is an $\mathcal{S}$-order-isomorphism and $(g_3 \circ g_4)(b) \in \mathrm{field}(\lfloor\leq\rfloor_{f(\beta)})$, i.e. $(g_3 \circ g_4)(b) < f(\beta)$. Then $(g_0 \circ g_1 \circ g_2 \circ g_3 \circ g_4){::}\lfloor\leq\rfloor_{f(\gamma)} \to \lfloor\leq\rfloor_{(g_3 \circ g_4)(b)}$ is an $\mathcal{S}$-order-isomorphism. By Lemma 3.32 we get $f(\gamma) = (g_3 \circ g_4)(b) < f(\beta)$, as was to be shown.

(2): Due to $X \in \mathcal{S}$, by Lemma 3.43, we have $\mathcal{SFO}(X)$, $\preceq_{\mathcal{SFO}(X)} \in \mathcal{S}$. Suppose that there is an injection $\pi : \mathcal{SFO}(X) \to X$ with $\pi \in \mathcal{S}$. Let us consider the reflexive ordering on $\mathrm{ran}(\pi)$ given by $\leq := \pi^{-1} \circ \preceq_{\mathcal{SFO}(X)} \circ \pi$, i.e. by $a \leq b$ if $\pi^{-1}(a) \preceq_{\mathcal{SFO}(X)} \pi^{-1}(b)$. Then $\pi{::}\preceq_{\mathcal{SFO}(X)} \to \leq$ is an $\mathcal{S}$-order-isomorphism by Lemma 3.42 and the Axiom of Simple Operations. Moreover, $\leq$ is an $\mathcal{S}$-well-ordering on a subclass of X. Set $\alpha := \mathcal{OT}(\leq, X)$. By Corollary 2.23, we have $\leq \in \alpha$. Thus, α is an $\mathcal{S}$-Frege ordinal. By Lemma 2.24, we have $\alpha \in \mathcal{S}$. Thus, $\alpha \in \mathcal{SFO}(X)$. For a *reductio ad absurdum*, suppose that there is an $\mathcal{S}$-order-isomorphism $f{::}\lfloor\preceq_{\mathcal{SFO}(X)}\rfloor_\alpha \to \leq$. Then there is also an $\mathcal{S}$-order-isomorphism $(f \circ \pi^{-1}){::}\lfloor\preceq_{\mathcal{SFO}(X)}\rfloor_\alpha \to \preceq_{\mathcal{SFO}(X)}$. This contradicts Lemma 3.31. **Q.e.d. (Lemma 3.45)**

Roughly speaking, Theorem 3.46 says that the cardinality of $\mathcal{S}$-FREGE ordinals over a given class X is neither smaller nor equal to the cardinality of X; cf. also Example 3.2.

Theorem 3.46 (HARTOGS' **Ordinal Theorem, [Hartogs, 1915]**)
Let us assume the Power-Set Axiom and the Axiom of Separation.
Moreover, let us assume $X \in \mathcal{S}$. *Then:*

(1) *For every* $\alpha \in \mathcal{SFO}(X)$ *and every* $\leq \in \alpha$,
the X-HARTOGS-*morphism onto* $\leq$ *is an* $\mathcal{S}$-*order-isomorphism.*

(2) *There is no injection* $\pi : \mathcal{SFO}(X) \to X$.

Proof of Theorem 3.46

(1): Let $\leq \in \alpha \in \mathcal{SFO}(X)$. Then $\leq$ is an $\mathcal{S}$-well-ordering on a subclass of X.
Let $f :: \lfloor \preceq_{\mathcal{SFO}(X)} \rfloor_\alpha \to \leq$ be the X-HARTOGS-morphism onto $\leq$.

By Lemma 3.45(1) and Lemma 2.17, it suffices to show $f \in \mathcal{S}$. By Lemma 3.43, we have $\mathcal{SFO}(X) \in \mathcal{S}$. By Corollary 1.25, we have $\mathrm{field}(\leq) \in \mathcal{S}$. Thus, by the Axiom of Simple operations (cf. Requirement 1.24), we have $f \subseteq \mathcal{SFO}(X) \times \mathrm{field}(\leq) \in \mathcal{S}$, i.e. $f \in \mathcal{S}$ by the Axiom of Separation.

(2): *Reductio ad absurdum.* Assume that $\pi : \mathcal{SFO}(X) \to X$ is an injective function. By Lemma 3.43, we have $\mathcal{SFO}(X) \in \mathcal{S}$. Thus, by the Axiom of Simple Operations, we have $\pi \subseteq \mathcal{SFO}(X) \times X \in \mathcal{S}$, i.e. $\pi \in \mathcal{S}$ by the Axiom of Separation. Thus, Lemma 3.45(2) contradicts Theorem 3.46(1). **Q.e.d. (Theorem 3.46)**

The following theorem shows that there is no chance to prove HARTOGS' Ordinal Theorem when we assume the Set Comprehension Axiom of ML.

Theorem 3.47 (Anti-HARTOGS**)**
Let us assume the Set Comprehension Axiom of ML. *Then*

(1) *Set* $\Omega := \mathcal{OT}(\preceq_{\mathcal{SFO}(\mathcal{S})}, \mathcal{S})$. *Then* $\Omega \in \mathcal{SFO}(\mathcal{S})$. *And for every* $\leq \in \Omega$,
the $\mathcal{S}$-HARTOGS-*morphism* $f :: \lfloor \preceq_{\mathcal{SFO}(\mathcal{S})} \rfloor_\Omega \to \leq$ *is an order-isomorphism,*
but, due to $f \notin \mathcal{S}$, *no* $\mathcal{S}$-*order-isomorphism.*

(2) $({}_{\mathcal{SFO}(\mathcal{S})}\upharpoonright \mathrm{id}) : \mathcal{SFO}(\mathcal{S}) \to \mathcal{S}$ *is an injection with* $({}_{\mathcal{SFO}(\mathcal{S})}\upharpoonright \mathrm{id}) \in \mathcal{S} \in \mathcal{S}$.

Proof of Theorem 3.47

(2): From the Set Comprehension Axiom of ML we get $\mathcal{S} \in \mathcal{S}$. Then we get $\mathcal{SFO}(\mathcal{S}) \in \mathcal{S}$ by Lemma 3.43. Thus, ${}_{\mathcal{SFO}(\mathcal{S})}\upharpoonright \mathrm{id} \in \mathcal{S}$ by the Axiom of Simple Operations. And trivially, id and ${}_{\mathcal{SFO}(\mathcal{S})}\upharpoonright \mathrm{id}$ are injective. Finally, we have $\mathcal{SFO}(\mathcal{S}) \subseteq \mathcal{S}$ by Corollary 3.35(2).

(1): From Theorem 3.47(2), by Lemma 3.45(2), we have $\preceq_{\mathcal{SFO}(\mathcal{S})} \in \Omega \in \mathcal{SFO}(X)$, but are assured that there is no $\mathcal{S}$-order-isomorphism $f :: \lfloor \preceq_{\mathcal{SFO}(\mathcal{S})} \rfloor_\Omega \to \preceq_{\mathcal{SFO}(\mathcal{S})}$. By Lemma 3.45(1), for every $\leq \in \Omega$, the $\mathcal{S}$-HARTOGS-morphism $f :: \lfloor \preceq_{\mathcal{SFO}(\mathcal{S})} \rfloor_\Omega \to \leq$ is an order-isomorphism, but cannot be an $\mathcal{S}$-order-isomorphism because $\leq$ is $\mathcal{S}$-order-isomorphic to $\preceq_{\mathcal{SFO}(\mathcal{S})}$. This means $f \notin \mathcal{S}$. **Q.e.d. (Theorem 3.47)**

*** END OF REVISION ***

Even in QUINE's ML class theory, from the assumption that such ordinals are sets, we can prove that the theory is inconsistent because then the class of FREGE ordinals contains a proper subclass without a minimal element. But in QUINE's NF this proof does not work because proper subclasses do not exist and only sub*sets* have to have a minimal element, and this way out can be chosen in ML as well.

4 Induction

We clearly need a section on Induction here. Also a section on the natural numbers would be nice, though not urgent.

Note that the following sections are not yet reworked according to the concretization on the theories of sets and classes of the previous sections!!!

5 Fixpoints of Monotonic and Expansive Functors

We now need the following notions for the first time, which we could have defined already in § 1.

Definition 5.1 (Monotonicity, Expansiveness, Fixpoint)
Let $\leq$ be a binary relation (on A). Let $f : A \to A$ be a singulary total function.
f is $\leq$-*monotonic* if $\forall x, y.\ \big(\ (x \leq y)\ \Rightarrow\ (f(x) \leq f(y))\ \big)$.
f is $\leq$-*expansive* if $\forall x \in A.\ (x \leq f(x))$.
s is a *fixpoint of* f if $s = f(s)$.

Definition 5.2 (Chain)
Let $\leq$ be a reflexive ordering.
C is a $\leq$-chain if $C \subseteq \mathrm{field}(\leq)$ and $\leq$ is total on C.
C is a *well-ordered* $\leq$-*chain* if $C \subseteq \mathrm{field}(\leq)$ and ${}_C{\restriction}{\leq}{\restriction}_C$ is a well-ordering on C.

5.1 Simple Construction of Greatest Fixpoint

The following theorem holds, even without assuming any (weak) forms of the Axiom of Choice or any kind of induction principle.

Theorem 5.3 (Knaster–Tarski)
Let $\leq$ be a reflexive ordering. Let f be $\leq$-monotonic.
If $\lambda x.\ (x \leq f(x))$ has a $\leq$-supremum $g := \sup^{\leq}_{x \leq f(x)} x$,
then g is the $\leq$-greatest fixpoint of f.

Proof of Theorem 5.3
For any x with $x{\leq}f(x)$ we have $x{\leq}g$ by (sup 1) of Definition 2.1, and then $x{\leq}f(x){\leq}f(g)$ by monotonicity. Thus, $g{\leq}f(g)$ by transitivity and (sup 2) of Definition 2.1. By monotonicity, $f(g){\leq}f(f(g))$. Thus, $f(g){\leq}g$ by (sup 1). Thus, $g{=}f(g)$ by anti-symmetry. Thus, g is a fixpoint of f, and by reflexivity and (sup 1) the $\leq$-greatest. **Q.e.d. (Theorem 5.3)**

5.2 Inductive Fixpoint Construction

The following is called ZERMELO*'s Fixpoint Theorem* after [Moschovakis, 2006, Note 18, p. 102], attributing the proof (but not the theorem) to [Zermelo, 1904]. We do not think this attribution to be justified, but there is no better name to the best of our knowledge.

Theorem 5.4 (ZERMELO**'s Fixpoint Theorem**)
Let $\leq$ be a reflexive ordering on a set A.
If the function $f : A \to A$ is $\leq$-expansive, and if each well-ordered $\leq$-chain has a $\leq$-supremum, then f has a fixpoint.

Proof of Theorem 5.4

Let $\preceq$ be the well-ordering on $\mathcal{FO}(A)$ given by HARTOGS' Theorem, cf. Theorem 3.46.
Let $\prec$ be the ordering of $\preceq$.

We define l_β for $\beta \in \mathcal{FO}(A)$ recursively over $\prec$ as follows:
If β is a non-limit $\preceq$-point (cf. Definition 3.5), we set $l_{\mathrm{S}_{\preceq}(\alpha)} := f(l_\alpha)$, for α being the $\preceq$-predecessor of β, cf. Lemma 3.6.
If β is a limit $\preceq$-point: $l_\beta := \sup^{\leq}_{\alpha\prec\beta} l_\alpha$.

Note that we will reuse this inductive construction in Theorem 5.5.

Claim 1: Let β be a non-limit $\preceq$-point and α its $\preceq$-predecessor. Then $l_\alpha \leq l_{\mathrm{S}_{\preceq}(\alpha)}$.
Proof of Claim 1: Directly by the presupposition that f is expansive. Q.e.d. (Claim 1)

Claim 2: In case of $\beta \prec \gamma$, we have $l_\beta \leq l_\gamma$.
Proof of Claim 2: If γ is a limit $\preceq$-point, this follows from (sup 1) of Definition 2.1. If γ is a non-limit ordinal $\mathrm{S}_{\preceq}(\alpha)$, this follows from Claim 1 in case of $\beta = \alpha$, and from induction hypothesis and Claim 1 in case of $\beta \prec \alpha$ by transitivity. Q.e.d. (Claim 2)

Note that the latter supremum in the recursive definition of l is indeed taken over well-ordered $\leq$-chains according to Claim 2: For $Q \subseteq \mathrm{ran}(l)$ with $Q \neq \emptyset$, we have $\min_{\leq} Q = l_\alpha$ for $\alpha := \min_{\preceq} \{\ \beta \in \mathcal{FO}(A) \mid l_\beta \in Q\ \}$.
From Claim 2 we can also show by induction that $\forall \beta \in \mathcal{FO}(A).\ \big(\ l_\beta = \sup^{\leq}_{\alpha\prec\beta} f(l_\alpha)\ \big)$, which gives an alternative definition of l.

Claim 3: If $l_\gamma \leq l_\beta$ for some $\gamma \succ \beta$, then $\forall \alpha \succ \beta.\ (l_\alpha = l_\beta)$.
Proof of Claim 3: In case of $\beta \prec \alpha \preceq \gamma$, we have $l_\beta \leq l_\alpha \leq l_\gamma$. Indeed, the first step holds by Claim 2 and the second by Claim 2 and reflexivity. By transitivity we get $l_\beta \leq l_\gamma$. By antisymmetry from the assumption we get $l_\alpha = l_\beta$.
The case of $\alpha \succ \gamma$ follows then by a trivial induction. Q.e.d. (Claim 3)

By HARTOGS' Theorem, the function l cannot be an injection. Thus, there must be $\gamma \succ \beta'$ with $l_\gamma = l_{\beta'}$. By reflexivity and Claim 3, we get $l_{\mathrm{S}_{\preceq}(\beta)'} = l_{\beta'}$, i.e. $f(l_{\beta'}) = l_{\beta'}$. Thus, $l_{\beta'}$ is a fixpoint of f. **Q.e.d. (Theorem 5.4)**

5.3 Inductive Construction of Least Fixpoint

Theorem 5.5 (Inductive Fixpoint Construction)
Let $\leq$ be a reflexive ordering on a set A and $f : A \to A$ be $\leq$-monotonic.
If each well-ordered $\leq$-chain has a $\leq$-supremum, then f has a $\leq$-least fixpoint x.
In addition, x actually satisfies $\forall y \in A.\ \big(\ (f(y) \leq y)\ \Rightarrow\ (x \leq y)\ \big)$.

Now if suprema *and* infima exist, due to duality we can choose between Theorem 5.3 and Theorem 5.5 for the construction of greatest and least fixpoints.

Proof of Theorem 5.5

Firstly, we show that exactly the same construction as in the Proof of Theorem 5.4 provides us with the same fixpoint, which is actually the least one due to monotonicity, according to Claim 4. Secondly, we sketch that Theorem 5.5 is actually a corollary of Theorem 5.4.

Let l be defined exactly as in the Proof of Theorem 5.4. Moreover, let also Claim 1, Claim 2, and Claim 3 be exactly as in the Proof of Theorem 5.4. Note that we have to give a new proof for Claim 1 only, because this is the only one that uses the presupposition of expansiveness.

New Proof of Claim 1: On the one hand, if α is a limit $\preceq$-point, then we have $l_\gamma \leq l_\alpha$ for all $\gamma \prec \alpha$ by definition of l_α and by (sup 1) of Definition 2.1. By induction hypothesis and monotonicity we then have $l_\gamma \leq l_{\mathrm{S}_\preceq(\gamma)} = f(l_\gamma) \leq f(l_\alpha) = l_{\mathrm{S}_\preceq(\alpha)}$. Then, by transitivity and (sup 2) of Definition 2.1, we have the claimed $l_\alpha \leq l_{\mathrm{S}_\preceq(\alpha)}$. On the other hand, if α is a non-limit $\preceq$-point $\mathrm{S}_\preceq(\gamma)$, then we have $l_\gamma \leq l_{\mathrm{S}_\preceq(\gamma)}$ by induction hypotheses and, by monotonicity, $l_\alpha = l_{\mathrm{S}_\preceq(\gamma)} = f(l_\gamma) \leq f(l_{\mathrm{S}_\preceq(\gamma)}) = f(l_\alpha) = l_{\mathrm{S}_\preceq(\alpha)}$. Q.e.d. (Claim 1)

Claim 4: Assume that $y \in A$ satisfies $f(y) \leq y$. For all $\alpha \in \mathcal{FO}(A)$ we have: $l_\alpha \leq y$.
Proof of Claim 4: On the one hand, if α is a limit $\preceq$-point, this follows from the induction hypothesis by (sup 2) of Definition 2.1. On the other hand, if α is a non-limit $\preceq$-point $\mathrm{S}_\preceq(\beta)$, then we have $l_\beta \leq y$ by induction hypothesis, and then by monotonicity: $l_\alpha = l_{\mathrm{S}_\preceq(\beta)} = f(l_\beta) \leq f(y) \leq y$. Q.e.d. (Claim 4)

Finally, let us sketch why Theorem 5.5 is actually a corollary to Theorem 5.4. Set

$$B \ := \ \big\{\ b \in A \ \big|\ \ b \leq f(b) \ \ \wedge\ \ \forall y \in A.\ \big(\ f(y) \leq y \ \Rightarrow\ b \leq y\ \big)\ \big\}.$$

Then B is closed under f and $\leq$-suprema. By the first property of the conjunction separating B from A we can now apply Theorem 5.4 to get a fixpoint. By the second property, this fixpoint is the least one and satisfies the additional property stated in Theorem 5.5. Cf. [Moschovakis, 2006, p. 103] for more details. **Q.e.d. (Theorem 5.5)**

6 Fixpoints of Class Operators

6.1 Class Operators

Definition 6.1 ([Monotonic] Class Operator)
Φ is a *class operator* if $\Phi(X)$ denotes a unique class for each class X.
A class operator Φ is called *monotonic* if it is monotonic for the subclass relation, i.e. if

$$\forall X, Y.\ \big(\ X \subseteq Y \Rightarrow\ \Phi(X) \subseteq \Phi(Y)\ \big).$$

Definition 6.2 ([Strongly] Set-Continuous)
A a class operator Φ is called *set-continuous* if

$$\forall X \subseteq \mathcal{V}.\ \left(\ \Phi(X) = \bigcup_{x \in \mathfrak{P}(X)} \Phi(x)\ \right).$$

Φ is *strongly set-continuous* if it additionally satisfies $\forall x \in \mathfrak{P}(\mathcal{V}).\ \big(\ \Phi(x) \in \mathfrak{P}(\mathcal{V})\ \big)$.

Corollary 6.3 *A class operator Φ is set-continuous iff it is monotonic and satisfies*

$$\forall X \not\in \mathcal{V}.\ \forall a \in \Phi(X).\ \exists x \in \mathfrak{P}(X).\ \big(\ a \in \Phi(x)\ \big)$$

Definition 6.4 (Algebraic)
A class operator Φ is *algebraic* if

$$\forall X \subseteq \mathcal{V}.\ \left(\ \Phi(X) = \bigcup_{x \in \mathfrak{P}_{\mathbf{N}}(X)} \Phi(x)\ \right).$$

Corollary 6.5 *An algebraic class operator is monotonic and set-continuous.*

6.2 Fixpoints of Set-Continuous Class Operators

Let Φ be a monotonic class operator. Then, by the KNASTER–TARSKI Theorem 5.3, $\bigcap_{X\supseteq\Phi(X)} X$ is the least and $\bigcup_{X\subseteq\Phi(X)} X$ the greatest fixpoint of Φ. Note that Theorem 5.5 cannot be applied to get a least or greatest fixpoint of Φ because its *set* A would have to be the superlarge collection of all classes.

6.2.1 The Least Fixpoint of a Set-Continuous Class Operator

For set-continuous class operators — and these are all monotonic class operators we are interested in — the following Theorem 6.8 is a real improvement over Theorem 5.5. It is a very minor refinement of Theorem 6.4 of [Aczel, 1988]. Not only does Theorem 6.8 require less set theory than Theorem 5.5, but also reduces the well-ordered construction of whole sets to the construction of their elements' derivation graphs, which is an interesting technique for showing properties of fixpoints.

Definition 6.6 ([Finitely Branching] Labeled Well-Founded Rooted Graphs)
A [*finitely branching*] *well-founded rooted graph* is a pair $(\longrightarrow, l)$, where l is a function, called *labeling function*, and $\longrightarrow$ is a binary relation on the set $\mathrm{dom}(l)$, such that $\longleftarrow$ is well-founded, $[\forall n \in \mathrm{dom}(l).\ \big(\ \langle\!\{n\}\!\rangle\!\longrightarrow\ \text{is finite}\ \big),]$ and such that there is a *root* $r \in \mathrm{dom}(l)$ with $\langle\!\{r\}\!\rangle\!\overset{*}{\longrightarrow}\ =\ \mathrm{dom}(l)$. As such a root is unique, we denote it with $\mathrm{root}(\longrightarrow)$.

By a trivial Noetherian induction over $\longleftarrow$ on the cardinality of $\langle\!\{u\}\!\rangle\!\overset{*}{\longrightarrow}$, we get the following corollary. Note that we do not need any weak form of the Axiom of Choice (such as König's Lemma) here, because we do not have to construct an infinite branch.

Corollary 6.7
If $(\longrightarrow, l)$ *is a finitely branching well-founded rooted graph, then* $\mathrm{dom}(l)$ *is finite.*

Theorem 6.8 *Let* Φ *be an* [*algebraic*] *set-continuous class operator.*
Let D *be the class of* [*finitely branching*] *labeled well-founded rooted graphs* $(\longrightarrow, l)$ *such that* $\forall i \in \mathrm{dom}(l).\ \big(\ l(i)\ \in\ \Phi(\{\ l(j)\ \mid\ i\!\longrightarrow\! j\ \})\ \big)$.
Set $I := \big\{\ \ l(\mathrm{root}(\longrightarrow))\ \big|\ (\longrightarrow, l) \in D\ \ \big\}$.
If Φ *is algebraic or if we assume the Axiom of Collection, then* I *is the least fixpoint of* Φ.

Note that we have

$$I = \bigcap_{X\supseteq\Phi(X)} X$$

in the case of Theorem 6.8, but the latter construction according to Theorem 5.3 requires a theory of sets *and classes*. For

$$I' := \bigcap_{x\supseteq\Phi(x)\ \wedge\ x\in\mathfrak{P}(\mathcal{V})} x$$

along the KNASTER–TARSKI proof we still get $I' \supseteq \Phi(I')$ and $\Phi(I') \supseteq \Phi(\Phi(I'))$. But $\Phi(I') \supseteq I'$ is guaranteed only if $\Phi(I') \in \mathfrak{P}(\mathcal{V})$, which (even if Φ is strongly set-continuous!) is not generally the case: To wit, consider NBG (where the class of cardinal numbers $\mathcal{C}$ is proper, Cantor's 2nd Diagonalization holds, and any proper class contains subsets of arbitrary cardinality) and define Φ by $\Phi(X) := \left\{ \begin{array}{ll} |\mathfrak{P}(X)| & \text{if } X \in \mathfrak{P}(\mathcal{V}) \\ \mathcal{C} & \text{otherwise} \end{array} \right\}$. By Corollary 6.3, Φ is strongly set-continuous, and we have $I = \mathcal{C}$ but $I' = \mathcal{V}$.

Proof of Theorem 6.8

Claim 1: $\Phi(I) \subseteq I$.
Proof of Claim 1: Assume $a \in \Phi(I)$. As Φ is set-continuous or algebraic, there is some $x \in \mathfrak{P}(I)$ or $x \in \mathfrak{P}_{\mathbf{N}}(I)$, resp., such that $a \in \Phi(x)$. Since $x \subseteq I$, by the definition of I we have
$$\forall y \in x.\ \exists(\longrightarrow, l).\ \big(\ (\longrightarrow, l) \in D\ \wedge\ y = l(\mathrm{root}(\longrightarrow))\ \big).$$
Since x is a set or a finite set, by the Axiom of Collection (cf. Definition 1.27) there is a set A or simply by induction on the size of x there is a finite set A, resp., such that
$$\forall y \in x.\ \exists(\longrightarrow, l) \in A.\ \big(\ (\longrightarrow, l) \in D\ \wedge\ y = l(\mathrm{root}(\longrightarrow))\ \big).$$
Define $B := \big\{\ \big(\ 0,\ ((\longrightarrow, l), i)\ \big)\ \big|\ (\longrightarrow, l) \in A \wedge i \in \mathrm{dom}(l)\ \big\}$. Note that the step from A to B is missing in the proof of Theorem 6.4 of [Aczel, 1988]. Now B is a set again, due to $B \subseteq \bigcup_{(\longrightarrow, l) \in A} \big(\ \{0\} \times (\{(\longrightarrow, l)\} \times \mathrm{dom}(l))\ \big)$. Then $C := B \uplus \{(1,0)\}$ is a set. Define the relation $\Longrightarrow$ to be the smallest relation on C such that, for each $(\longrightarrow, l) \in A$, we have
$$(1,0) \Longrightarrow (0, ((\longrightarrow, l), \mathrm{root}(\longrightarrow)))$$
and
$$\forall i, j \in \mathrm{dom}(l).\ \big(\ j \longrightarrow i\quad \Rightarrow\quad (0, ((\longrightarrow, l), j)) \Longrightarrow (0, ((\longrightarrow, l), i))\ \big).$$
We define L on C by $L(0, ((\longrightarrow, l), i)) := l(i)$ and $L(1,0) := a$. Now the second following =-step is the one that requires the definition of B in addition to A: We have
$$\begin{array}{lll} & L(0, ((\longrightarrow, l), i)) & \\ = & l(i)\ \in\ \Phi(\{\ l(j) \mid i \longrightarrow j\ \}) & =\ \Phi\big(\ \big\{\ l(j)\ \big|\ (0, ((\longrightarrow, l), i)) \Longrightarrow (0, ((\longrightarrow, l), j))\ \big\}\ \big) \\ & & =\ \Phi\big(\ \big\{\ L(J)\ \big|\ (0, ((\longrightarrow, l), i)) \Longrightarrow J\ \big\}\ \big), \end{array}$$
and by $x \subseteq \{\ l(\mathrm{root}(\longrightarrow)) \mid (\longrightarrow, l) \in A\ \}$ and monotonicity of Φ we have
$$\begin{array}{rl} L(1,0) = a \in \Phi(x) & \subseteq\ \Phi\big(\ \big\{\ l(\mathrm{root}(\longrightarrow))\ \big|\ (\longrightarrow, l) \in A\ \big\}\ \big) \\ & =\ \Phi\big(\ \big\{\ L(J)\ \big|\ (1,0) \Longrightarrow J\ \big\}\ \big) \end{array}.$$
All in all, $(\Longrightarrow, L) \in D$, i.e. $a = L(1,0) = L(\mathrm{root}(\Longrightarrow)) \in I$. Q.e.d. (Claim 1)

Claim 2: If $\Phi(X) \subseteq X$, then $I \subseteq X$.
Proof of Claim 2: Let $(\longrightarrow, l) \in D$. We have to show $l(\mathrm{root}(\longrightarrow)) \in X$. It suffices to show $l(i) \in X$ for $i \in \mathrm{dom}(l)$ by induction on $\longleftarrow$. By induction hypothesis: $\{\ l(j) \mid i \longrightarrow j\ \} \subseteq X$. By definition of D and monotonicity, we have
$l(i) \in \Phi(\{\ l(j) \mid i \longrightarrow j\ \}) \subseteq \Phi(X) \subseteq X$. Q.e.d. (Claim 2)

By Claim 1 and monotonicity we have $\Phi(\Phi(I)) \subseteq \Phi(I) \subseteq I$. Thus, by Claim 2, $I \subseteq \Phi(I)$. By antisymmetry of $\subseteq$, I is a fixpoint of Φ, and by Claim 2 the least one.

Q.e.d. (Theorem 6.8)

6.2.2 The Greatest Fixpoint of a Set-Continuous Class Operator

The following is a theorem of set theory.

Theorem 6.9 ([Aczel, 1988], Theorem 6.5) *Let $\varPhi$ be some set-continuous class operator. Assume the Axiom of Collection and the Principle of Dependent Choice. Set*

$$J := \bigcup_{x \subseteq \varPhi(x) \ \wedge \ x \in \mathfrak{P}(\mathcal{V})} x.$$

Now: J is the greatest fixpoint of $\varPhi$.

Note that we have

$$J = \bigcup_{X \subseteq \varPhi(X)} X$$

in the case of Theorem 6.9, but the latter construction according to Theorem 5.3 requires a theory of sets *and classes*.

Example 6.10 (Set-Continuity Necessary in Theorem 6.9)
Along the KNASTER–TARSKI proof we always get $J \subseteq \varPhi(J) \subseteq \varPhi(\varPhi(J))$. But $J \supseteq \varPhi(J)$ is guaranteed only if $\varPhi(J) \in \mathfrak{P}(\mathcal{V})$, which is not generally the case: To wit, consider NBG (where a subclass of a set is always a set) and define $\varPhi$ by $\varPhi(X) := \left\{ \begin{array}{ll} X \cap \mathcal{O} & \text{if } X \in \mathfrak{P}(\mathcal{V}) \\ \mathcal{V} & \text{otherwise.} \end{array} \right\}$.
Then $\varPhi$ is monotonic (but not set-continuous), and we have $J = \mathcal{O}$ but $\varPhi(J) = \mathcal{V} = \bigcup_{X \subseteq \varPhi(X)} X$. Thus, $J \not\supseteq \varPhi(J)$.

Example 6.11 (No Analog of Theorem 6.9 for Algebraic $\varPhi$ as for Theorem 6.8)
Define $\varPhi(X) := \{\ y \in \mathcal{O} \mid \exists x \in X.\ y \prec x\ \}$. Then $\varPhi$ is algebraic: $\varPhi(X) = \bigcup_{\exists y \in X.\ x = \{y\}} \varPhi(x)$.
We have $J = \mathcal{O}$, but $\bigcup_{x \subseteq \varPhi(x) \ \wedge \ x \in \mathfrak{P}_{\mathbf{N}}(\mathcal{V})} \varPhi(x) = \emptyset$.

Proof of Theorem 6.9

Claim 1: $J \subseteq \varPhi(J)$.
Proof of Claim 1: Let $a \in J$. Then there is some $x \in \mathcal{S}$ with $x \subseteq \varPhi(x)$ and $a \in x$. Then $x \subseteq J$. Then $a \in x \subseteq \varPhi(x) \subseteq \varPhi(J)$ by monotonicity. Q.e.d. (Claim 1)

Claim 2: If $X \subseteq \varPhi(X)$ and $x \in \mathfrak{P}(X)$, then there is an $x' \in \mathfrak{P}(X)$ such that $x \subseteq \varPhi(x')$.
Proof of Claim 2: We have $x \subseteq X \subseteq \varPhi(X)$. As $\varPhi$ is set-continuous, this means

$$\forall y \in x.\ \exists u.\ \big(\ u \in \mathfrak{P}(X) \wedge y \in \varPhi(u)\ \big).$$

Since x is a set, by the Axiom of Collection there is a set A such that

$$\forall y \in x.\ \exists u \in A.\ \big(\ u \in \mathfrak{P}(X) \wedge y \in \varPhi(u)\ \big).$$

Set $A' := A \cap \mathfrak{P}(X)$ and $x' := \bigcup A'$. Then x' is a set with $x' \subseteq X$, and, by monotonicity, we have $\varPhi(u) \subseteq \varPhi(x')$ for any $u \in A'$. Thus, $x \subseteq \bigcup_{u \in A'} \varPhi(u) \subseteq \varPhi(x')$. Q.e.d. (Claim 2)

Claim 3: If $X \subseteq \varPhi(X)$, then $X \subseteq J$.
Proof of Claim 3: Assume $a \in X$. By Claim 2 and the Principle of Dependent Choice, there is an $x : \mathbf{N} \to \mathfrak{P}(X)$ with $x_0 = \{a\}$ and $\forall i \in \mathbf{N}.\ x_i \subseteq \varPhi(x_{i+1})$. Set $z := \bigcup_{i \in \mathbf{N}} x_i$. Then $a \in z \in \mathfrak{P}(X)$. Moreover, by monotonicity, $z = \bigcup_{i \in \mathbf{N}} x_i \subseteq \bigcup_{i \in \mathbf{N}} \varPhi(x_{i+1}) \subseteq \varPhi(z)$. Thus, $a \in z \subseteq J$. Q.e.d. (Claim 3)

By Claim 1 and monotonicity we have $J \subseteq \varPhi(J) \subseteq \varPhi(\varPhi(J))$. Thus, by Claim 3, $\varPhi(J) \subseteq J$. By antisymmetry of $\subseteq$, J is a fixpoint of $\varPhi$, and by Claim 3 the greatest one.

Q.e.d. (Theorem 6.9)

7 Lattices

Definition 7.1 (Compactly Generated)

Let $\leq$ be a binary relation (on A).

An element a (of A) is $\leq$-*compact* if for every $\leq$-supremum s (in A) of every singulary predicate P (on A) with $a{\leq}s$, there are finitely many $y_1, \ldots, y_n$ (in A) ($n \in \mathbf{N}$), such that $\forall i \in \{1, \ldots, n\}.\ P(y_i)$ and such that $\lambda x.\ (x \in \{y_1, \ldots, y_n\})$ has a $\leq$-supremum with $a \leq \sup^{\leq}\{y_1, \ldots, y_n\}$.

An element c (of A) is $\leq$-*compactly generated* if c is a $\leq$-supremum of some predicate P with $\forall x.\ \big(\ P(x) \Rightarrow (x \text{ is } \leq\text{-compact})\ \big)$.

Corollary 7.2 *Let $\leq$ be a binary relation (on A). Let $a \in A$. If a is $\leq$-compact, then a is $\leq$-compactly generated.*

Definition 7.3 ([Algebraic] [Complete] Lattice)
A *lattice* (on A) is a reflexive ordering $\leq$ (on A) where any predicate P (on A) with $\exists x, y.\ \forall z.\ (P(z) \Leftrightarrow z{=}x \vee z{=}y)$ has an $\leq$-supremum and an $\leq$-infimum (in A).
A *complete lattice* (on A) is a reflexive ordering $\leq$ (on A) where any singulary predicate (on A) has an $\leq$-supremum (in A).
An *algebraic lattice* (on A) is a complete lattice (on A), where every element (of A) is $\leq$-compactly generated.

Corollary 7.4 *Let $\leq$ be a complete lattice on $A \subseteq \mathrm{V}$. Set $A' := \{\ a \in A \mid a \text{ is } \leq\text{-compact}\ \}$. Let $c \in A$. Then c is $\leq$-compactly generated iff $c = \sup^{\leq}\{\ a \in A' \mid a \leq c\ \}$.*

The following is a corollary of Lemma 2.2 and Corollary 2.13:

Corollary 7.5
A complete lattice is a lattice; and the dual of a [complete] lattice is a [complete] lattice.

Lemma 7.6 *If $\leq$ and $\preceq$ are two complete lattices, and $f{::}{\leq}{\to}{\preceq}$ is an order-isomorphism, then*

$$\forall X \subseteq \mathrm{field}(\leq).\ \left(\begin{array}{cc} & f(\inf^{\leq} X) = \inf^{\preceq}(\langle X\rangle f) \\ \wedge & f(\sup^{\leq} X) = \sup^{\preceq}(\langle X\rangle f) \end{array}\right).$$

Proof of Lemma 7.6 For $x \in X$, we have $x \leq \sup^{\leq} X$. Thus, $f(x) \preceq f(\sup^{\leq} X)$. This means that $f(\sup^{\leq} X)$ satisfies (sup 1) of Definition 2.1 for the supremum of $\lambda y.\ (y \in \langle X\rangle f)$. It remains to show that it also satisfies (sup 2). Thus, suppose that, for some y, we have $f(x) \preceq y$ for all $x \in X$. As f is surjective on field($\leq'$), there is some $z \in \mathrm{field}(\leq)$ with $y{=}f(z)$. Then $f(x) \preceq f(z)$. Then $x \leq z$ for all $x \in X$. Then $\sup^{\leq} X \leq z$. Then $f(\sup^{\leq} X) \preceq f(z) = y$, as was to be shown. **Q.e.d. (Lemma 7.6)**

Corollary 7.7 *If $\leq$ and $\preceq$ are two lattices, and $f{::}{\leq}{\to}{\preceq}$ is an order-isomorphism, then*

$$\forall x, y \in \mathrm{field}(\leq).\ \left(\begin{array}{cc} & f(\inf^{\leq}\{x, y\}) = \inf^{\preceq}\{f(x), f(y)\} \\ \wedge & f(\sup^{\leq}\{x, y\}) = \sup^{\preceq}\{f(x), f(y)\} \end{array}\right).$$

8 Closure Operators

The following is based on [Burris & Sankappanavar, 1981, Chapter I, § 5].

Definition 8.1 ([Algebraic] Closure Operator)
An [*algebraic*] *closure operator* is an [algebraic] monotonic class operator C satisfying
$$\forall X{\subseteq}\mathcal{V}.\ \big(\ X \cup C(C(X)) \subseteq C(X)\ \big).$$

Remark 8.2 (Dual Closure Operator)
Note that the dual concept of a monotonic class operator C' satisfying
$$\forall X{\subseteq}\mathcal{V}.\ \big(\ X \cap C'(C'(X)) \supseteq C'(X)\ \big)$$
does not seem to be needed at first glance, because we can turn C' into a closure operator C simply by defining
$$C(X) := \mathcal{V} \setminus C'(\mathcal{V}\setminus X).$$
We make use of this trick in Definition 10.4 for the greatest-fixpoint operator, which would naturally satisfy the dual concept. But cf. Remark 10.7.

Lemma 8.3 *Let C be a closure operator.*

1. *X is a fixpoint of C iff $X = C(Y)$ for some $Y \subseteq \mathcal{V}$.*
2. *For a predicate P holding only for classes (not necessarily fixpoints):*
   (a) $C\big(\ \bigcup_{P(Z)} C(Z)\ \big) = C\big(\ \bigcup_{P(Z)} Z\ \big)$.
   (b) $C\big(\ \bigcap_{P(Z)} C(Z)\ \big) = \bigcap_{P(Z)} C(Z)$.

Proof of Lemma 8.3

1: On the one hand, if X is a fixpoint of C, then $X = C(X)$. On the other hand, we have both $C(C(Y)) \subseteq C(Y)$ and $C(Y) \subseteq C(C(Y))$, and thus $C(Y) = C(C(Y))$, i.e. $C(Y)$ is a fixpoint of C.

2a"$\subseteq$": For every class Z' with $P(Z')$ we have $Z' \subseteq \bigcup_{P(Z)} Z$. By monotonicity $C(Z') \subseteq C(\bigcup_{P(Z)} Z)$. Thus $\bigcup_{P(Z')} C(Z') \subseteq C(\bigcup_{P(Z)} Z)$. By monotonicity $C(\bigcup_{P(Z')} C(Z')) \subseteq C(C(\bigcup_{P(Z)} Z)) \subseteq C(\bigcup_{P(Z)} Z)$.

2a"$\supseteq$": By extensiveness, we have $Z \subseteq C(Z)$ for every Z. Thus, $\bigcup_{P(Z)} Z \subseteq \bigcup_{P(Z)} C(Z)$. By monotonicity $C(\bigcup_{P(Z)} Z) \subseteq C(\bigcup_{P(Z)} C(Z))$.

2b"$\subseteq$": For every class Z' with $P(Z')$ we have $\bigcap_{P(Z)} C(Z) \subseteq C(Z')$. By monotonicity $C(\bigcap_{P(Z)} C(Z)) \subseteq C(C(Z')) \subseteq C(Z')$. Thus $C(\bigcap_{P(Z)} C(Z)) \subseteq \bigcap_{P(Z')} C(Z')$.

2b"$\supseteq$": By extensiveness.

Q.e.d. (Lemma 8.3)

9 Fixpoint Lattices

As corollary from Lemma 8.3 we get:

Corollary 9.1 (Fixpoint Lattice)
The subclass relation restricted to the fixpoints of a closure operator C forms a complete lattice, where, for any predicate P holding only for such fixpoints,

$$\inf^{\subseteq}_{P(X)} X = \bigcap_{P(X)} X$$

and

$$\sup^{\subseteq}_{P(X)} X = C\big(\bigcup_{P(X)} X \big).$$

This lattice is called the fixpoint lattice of C.

Lemma 9.2 (Algebraic Fixpoint Lattice)
If C is an algebraic closure operator, then the fixpoint lattice of C is an algebraic lattice and its compact elements are precisely the finitely generated fixpoints, i.e. the $C(Y)$ with $Y \in \mathfrak{P}_{\mathbf{N}}(\mathcal{V})$.

Proof of Lemma 9.2

Claim 1: $C(\{x_1, \ldots, x_m\})$ is compact for $m \in \mathbf{N}$.
Proof of Claim 1: Let us assume $C(\{x_1, \ldots, x_m\}) \subseteq \sup_{P(Z)} C(Z)$.
According to Lemma 8.3(2a), our assumption is $C(\{x_1, \ldots, x_m\}) \subseteq C\big(\bigcup_{P(Z)} Z \big)$.
By Definition 8.1, for each $i \in \{1, \ldots, m\}$, we have

$$\{x_i\} \subseteq C(\{x_i\}) \subseteq C(\{x_1, \ldots, x_m\}) \subseteq C\big(\bigcup_{P(Z)} Z \big).$$

By Definition 6.4, there is some finite set $X_i \subseteq \bigcup_{P(Z)} Z$ with $x_i \in C(X_i)$. Thus there is some $o_i \in \mathbf{N}$ such that $\forall k \in \{1, \ldots, o_i\}.\ P(Z_{i,k})$ and $X_i \subseteq \bigcup_{k\in\{1,\ldots,o_i\}} Z_{i,k}$. By monotonicity of C, we have $\{x_i\} \subseteq C\big(\bigcup_{k\in\{1,\ldots,o_i\}} Z_{i,k} \big)$. By monotonicity again and Lemma 8.3(2a), $C(\{x_1, \ldots, x_m\}) \subseteq C\Big(\bigcup_{i\in\{1,\ldots,n\}} C\big(\bigcup_{k\in\{1,\ldots,o_i\}} Z_{i,k} \big) \Big) = C\big(\bigcup_{i\in\{1,\ldots,n\}} \bigcup_{k\in\{1,\ldots,o_i\}} Z_{i,k} \big) = C\big(\bigcup_{i\in\{1,\ldots,n\},\ k\in\{1,\ldots,o_i\}} C(Z_{i,k}) \big) = \sup_{i\in\{1,\ldots,n\},\ k\in\{1,\ldots,o_i\}} C(Z_{i,k})$ and $\forall i \in \{1, \ldots, n\}.\ \forall k \in \{1, \ldots, o_i\}.\ P(Z_{i,k})$. This was to be shown according to Definition 7.1. Q.e.d. (Claim 1)

Claim 2: For any $X \subseteq \mathcal{V}$ we have $C(X) = \bigcup_{x\in\mathfrak{P}_{\mathbf{N}}(X)} C(x) = \sup_{x\in\mathfrak{P}_{\mathbf{N}}(X)} C(x)$.
Proof of Claim 2: The first equation is just the definition of an algebraic class operator, cf. Definition 6.4. The second follows by applying C to both sides of the first equation. $C(X) = C\big(C(X) \big) = C\big(\bigcup_{x\in\mathfrak{P}_{\mathbf{N}}(X)} C(x) \big) = \sup_{x\in\mathfrak{P}_{\mathbf{N}}(X)} C(x)$ by Definition 8.1 and Corollary 9.1, respectively. Q.e.d. (Claim 2)

Claim 3: If $C(X)$ is compact, then it is finitely generated.
Proof of Claim 3: By Claim 2, $C(X) = \sup_{x\in\mathfrak{P}_{\mathbf{N}}(X)} C(x)$. If $C(X)$ is compact, then, by Definition 7.1, there is some $Y \in \mathfrak{P}_{\mathbf{N}}(\mathfrak{P}_{\mathbf{N}}(X))$ with $C(X) \subseteq \sup_{x\in Y} C(x) = C\big(\bigcup_{x\in Y} x \big)$ by Corollary 9.1 and Lemma 8.3(2a). Q.e.d. (Claim 3)

By Claim 1 and Claim 3 the compact elements are precisely the finitely generated fixpoints. Thus, according to Claim 2, Definitions 7.1 and 7.3, the fixpoint lattice of C is an algebraic lattice. **Q.e.d. (Lemma 9.2)**

Definition 9.3 (General-Closure, Compact-Closure, Algebraic-Closure)
Let $\leq$ be a complete lattice on $A \subseteq \mathcal{V}$. Let $A' := \{\ a \in A \mid a \text{ is } \leq\text{-compact}\ \}$.

1. The *general-closure operator of* $\leq$ is
$$C(X) \quad := \quad \{\ y \in A \mid y \leq \sup^{\leq}(X \cap A)\ \} \uplus X \backslash A.$$
2. The *compact-closure operator of* $\leq$ is
$$C'(X) \quad := \quad \{\ y \in A' \mid y \leq \sup^{\leq}(X \cap A)\ \} \uplus X \backslash A'.$$
3. The *algebraic-closure operator of* $\leq$ is
$$C''(X) \quad := \quad \{\ y \in A' \mid y \leq \sup^{\leq}(X \cap A')\ \} \uplus X \backslash A'.$$

Corollary 9.4
Let $\leq$ be a complete lattice on $A \subseteq \mathcal{V}$. Set $A' := \{\ a \in A \mid a$ is $\leq$-compact $\}$.
Let C' be the compact-closure operator of $\leq$.
Let C'' be the algebraic-closure operator of $\leq$.
Then, for $X \subseteq A'$, we have $C'(X) = C''(X)$.

Lemma 9.5
If $\leq$ is a complete lattice on $A \subseteq \mathcal{V}$, then the general-closure operator of $\leq$ is indeed a closure operator, and the compact-closure operator and the algebraic-closure operator of $\leq$ are algebraic closure operators.

Proof of Lemma 9.5

Set $A' := \{\ a \in A \mid a \text{ is } \leq\text{-compact}\ \}$. The general-closure operator C, the compact-closure operator C', and the algebraic-closure operator C'' of $\leq$ are monotonic class operators by simple inspection. Moreover, they are obviously $\subseteq$-extensive. Note that requirement of extensiveness forces us to include $X \backslash A$ or $X \backslash A'$, respectively.

Claim 1: We have $C(C(X)) \subseteq C(X)$, $C'(C'(X)) \subseteq C'(X)$, and $C''(C''(X)) \subseteq C''(X)$.
Proof of Claim 1: For $x \in C(X) \setminus X$, we have $x \leq \sup^{\leq}(X \cap A)$. Thus, $\sup^{\leq}(C(X) \cap A) \leq \sup^{\leq}(X \cap A)$. Thus, $C(C(X)) \subseteq C(X)$. For $x \in C'(X) \setminus X$ or $x \in C''(X) \setminus X$, we have $x \leq \sup^{\leq}(X \cap A)$ or $x \leq \sup^{\leq}(X \cap A')$, respectively. Thus, $\sup^{\leq}(C'(X) \cap A) \leq \sup^{\leq}(X \cap A)$ and $\sup^{\leq}(C''(X) \cap A') \leq \sup^{\leq}(X \cap A')$. Thus, $C'(C'(X)) \subseteq C'(X)$ and $C''(C''(X)) \subseteq C''(X)$. Q.e.d. (Claim 1)

Claim 2: C' and C'' are algebraic.
Proof of Claim 2: Assume $X \subseteq \mathcal{V}$, and $x \in C'(X)$ or $x \in C''(X)$, respectively. Then we have to find $Y \in \mathfrak{P}_{\mathbf{N}}(X)$ such that $x \in C'(Y)$ or $x \in C''(Y)$, respectively. If $x \in \mathcal{V} \backslash A'$, then we can take $Y := \{x\}$. Thus, we may assume $x \in A'$. Then we have $x \leq \sup^{\leq}(X \cap A)$ or $x \leq \sup^{\leq}(X \cap A')$, respectively. As x is $\leq$-compact (cf. Definition 7.1), there must be some Y such that $Y \in \mathfrak{P}_{\mathbf{N}}(X \cap A)$ or $Y \in \mathfrak{P}_{\mathbf{N}}(X \cap A')$, resp., and $x \leq \sup^{\leq} Y$. Then $x \leq \sup^{\leq}(Y \cap A)$ or $x \leq \sup^{\leq}(Y \cap A')$, respectively. Then $x \in C'(Y)$ or $x \in C''(Y)$, respectively. Q.e.d. (Claim 2)

Q.e.d. (Lemma 9.5)

By Corollary 9.1, every closure operator gives rise to a complete lattice, namely to its fixpoint lattice. By Lemma 9.5, every complete lattice gives rise to a closure operator, namely to its general-closure operator. As the following lemma states, this closure operator gives a representation of the original lattice up to isomorphism.

Lemma 9.6 (Complete Lattice = Fixpoint Lattice of General-Closure)
Let $\leq$ be a complete lattice on $A \subseteq \mathcal{V}$. Let C be the general-closure operator of $\leq$.
Let $\preceq$ be the fixpoint lattice of C restricted to those fixpoints Y with $Y \subseteq A$.
Let $f : A \to \mathfrak{P}(A)$ be given by $f(a) := C(\{a\})$.
Then $f{::}{\leq}{\to}{\preceq}$ is an order-isomorphism with inverse $X \mapsto \sup^{\leq} X$ ($X \subseteq A$ with $C(X){=}X$).

Proof of Lemma 9.6
Firstly, we have to show that the restriction of the fixpoint lattice of C to those fixpoints Y with $Y \subseteq A$ is a complete lattice again. Indeed, it is actually a complete sublattice because, according to Corollary 9.1, for P holding only for such fixpoints, we have $\sup^{\subseteq}_{P(X)} X = C(\bigcup_{P(X)} X) = \{\, y \in A \mid y \leq \sup^{\leq}((\bigcup_{P(X)} X) \cap A) \,\} \uplus (\bigcup_{P(X)} X) \setminus A = \{\, y \in A \mid y \leq \sup^{\leq}(\bigcup_{P(X)} X) \,\} \subseteq A$. Secondly, for $a \in A$ we have $f(a) = C(\{a\}) = \{\, y \in A \mid y \leq \sup^{\leq}(\{a\} \cap A) \,\} \uplus \{a\} \setminus A = \{\, y \in A \mid y \leq \sup^{\leq}(\{a\}) \,\} = \{\, y \in A \mid y \leq a \,\}$. Thus, we have $\sup^{\leq}(f(a)) = a$, and $f : A \to \mathrm{field}(\preceq)$ is bijective with the given inverse, and we obviously have $\forall a_0, a_1 \in A.\ \big(\ (a_0 \leq a_1) \Leftrightarrow (f(a_0) \subseteq f(a_1))\ \big)$. **Q.e.d. (Lemma 9.6)**

By Lemma 9.2, every algebraic closure operator gives rise to an algebraic lattice, namely to its fixpoint lattice. By Lemma 9.5, every complete lattice gives rise to two algebraic closure operators, namely to its compact-closure operator and to its algebraic-closure operator. As the following lemma states, each of these closure operators gives a representation of the original lattice up to isomorphism, provided that this original lattice is actually algebraic.

Lemma 9.7 (Algebraic Lattice = Fixpoint Lattice of Compact-Closure)
Let $\leq$ be an algebraic lattice on $A \subseteq \mathcal{V}$. Set $A' := \{\, a \in A \mid a \text{ is } \leq\text{-compact} \,\}$.
Let C' be the compact-closure operator of $\leq$.
Let C'' be the algebraic-closure operator of $\leq$.
Let $\preceq$ be the fixpoint lattice of C' restricted to those fixpoints Y with $Y \subseteq A'$.

1. *$\preceq$ is identical to the fixpoint lattice of C'' restricted to those fixpoints Y with $Y \subseteq A'$.*
2. *Let $f : A \to \mathfrak{P}(A)$ be given by $f(a) := C'(\{a\})$. Then $f{::}{\leq}{\to}{\preceq}$ is an order-isomorphism with inverse $X \mapsto \sup^{\leq} X$ ($X \subseteq A'$ with $C(X){=}X$).*

Note that the reason for the compact-closure (or algebraic-closure) operator in addition to the general-closure operator is that — in general — the general-closure operator of an algebraic lattice does not seem to be algebraic, and the compact-closure of a complete but non-algebraic lattice does not provide an isomorphism.

Proof of Lemma 9.7

<u>1:</u> By Corollary 9.4.

<u>2:</u> Firstly, we have to show that the restriction of the fixpoint lattice of C' to those fixpoints Y with $Y \subseteq A'$ is an algebraic lattice again. Indeed, it is actually a complete sublattice because, according to Corollary 9.1, for P holding only for such fixpoints, we have $\sup^{\subseteq}_{P(X)} X = C'(\bigcup_{P(X)} X) = \{\, y \in A' \mid y \leq \sup^{\leq}((\bigcup_{P(X)} X) \cap A) \,\} \uplus (\bigcup_{P(X)} X) \setminus A'$ $= \{\, y \in A' \mid y \leq \sup^{\leq}(\bigcup_{P(X)} X) \,\} \subseteq A'$. Secondly, for $a \in A'$ we have $f(a) = C'(\{a\}) = \{\, y \in A' \mid y \leq \sup^{\leq}(\{a\} \cap A) \,\} \uplus \{a\} \setminus A' =$ $\{\, y \in A' \mid y \leq \sup^{\leq}(\{a\}) \,\} = \{\, y \in A' \mid y \leq a \,\}$. Thus, as a is $\leq$-compactly generated, we have $\sup^{\leq}(f(a)) = a$ by Corollary 7.4, and $f : A \to \mathrm{field}(\preceq)$ is bijective with the given inverse, and we obviously have $\forall a_0, a_1 \in A.\ \big(\ (a_0 \leq a_1) \Leftrightarrow (f(a_0) \subseteq f(a_1))\ \big)$.

Q.e.d. (Lemma 9.7)

10 Closure Operators from Monotonic Class Operators

Definition 10.1 (LC)
The *least-fixpoint operator* LC maps any class operator $\varPhi$ to a class operator $\mathrm{LC}(\varPhi)$ given by $\mathrm{LC}(\varPhi)(X) := \bigcap_{Z \supseteq \varPhi(Z) \cup X} Z$ for $X \subseteq \mathcal{V}$.

Lemma 10.2 *If $\varPhi$ is a monotonic class operator, then* $\mathrm{LC}(\varPhi)$ *is a closure operator which satisfies* $\mathrm{LC}(\varPhi)(X) = \varPhi(\mathrm{LC}(\varPhi)(X)) \cup X$ *for all $X \subseteq \mathcal{V}$.*

Proof of Lemma 10.2
Let $\varPhi$ be a monotonic class operator. Define $\varPhi_\cup$ by $\varPhi_\cup(X)(Y) := \varPhi(Y) \cup X$. Then $\varPhi_\cup(X)$ is a monotonic class operator as well, for every $X \subseteq \mathcal{V}$. Then, by Theorem 5.3 applied to $\varPhi_\cup(X)$, we get that $\mathrm{LC}(\varPhi)(X)$ is the least fixpoint of $\varPhi_\cup(X)$. Thus, $\mathrm{LC}(\varPhi)$ satisfies the equation stated in the lemma. Thus, $\mathrm{LC}(\varPhi)$ is extensive. Moreover, $\mathrm{LC}(\varPhi)$ is obviously a monotonic class operator. It remains to show $\mathrm{LC}(\varPhi)\big(\ \mathrm{LC}(\varPhi)(X)\ \big) \subseteq \mathrm{LC}(\varPhi)(X)$ for every $X \subseteq \mathcal{V}$. This means that we have to show $\mathrm{LC}(\varPhi)\big(\ V\ \big) \subseteq W$ for $V := \mathrm{LC}(\varPhi)(X)$ and $W := \mathrm{LC}(\varPhi)(X)$. This means to show $\bigcap_{Z \supseteq \varPhi(Z) \cup V} Z \subseteq W$. For this it again suffices to show $W \supseteq \varPhi(W) \cup V$. But $W \supseteq \varPhi(W)$ has already been shown and $W \supseteq V$ is trivial. **Q.e.d. (Lemma 10.2)**

Theorem 10.3
Let $\varPhi$ be a monotonic class operator.
If $\varPhi$ is algebraic or if we assume the Axiom of Collection, then we have the following:

1. *If $\varPhi$ is set-continuous, then* $\mathrm{LC}(\varPhi)$ *is set-continuous, too.*
2. *If $\varPhi$ is algebraic, then* $\mathrm{LC}(\varPhi)$ *is algebraic, too.*

Proof of Theorem 10.3
Let $\varPhi_\cup$ be given as in the Proof of Lemma 10.2. Assume $X \subseteq \mathcal{V}$ and $a \in \mathrm{LC}(\varPhi)(X)$. As $\mathrm{LC}(\varPhi)$ is a monotonic class operator, by Corollary 6.3, we only have to find some $x \in \mathfrak{P}(X)$ (or even $x \in \mathfrak{P}_{\mathbf{N}}(X)$) with $a \in \mathrm{LC}(\varPhi)(x)$. As $\varPhi$ is set-continuous (or even algebraic), $\varPhi_\cup(X)$ is set-continuous (or even algebraic), too. Thus, as $\mathrm{LC}(\varPhi)(X)$ is the least fixpoint of $\varPhi_\cup(X)$ by Lemma 10.2 and Definition 10.1, according to Theorem 6.8, we have $a = l(\mathrm{root}(\longrightarrow))$ for a [finitely branching] well-founded rooted graph $(\longrightarrow, l)$ with

$$\forall i \in \mathrm{dom}(l).\ \big(\ l(i) \in \varPhi_\cup(X)(\{\ l(j) \mid i \longrightarrow j\ \})\ \big).$$

Set $x := \mathrm{ran}(l) \cap X$. Then $x \in \mathfrak{P}(X)$ (or even $x \in \mathfrak{P}_{\mathbf{N}}(X)$ by Corollary 6.7). By definition of x and $\varPhi_\cup$ we have

$$\forall i \in \mathrm{dom}(l).\ \big(\ l(i) \in \varPhi_\cup(x)(\{\ l(j) \mid i \longrightarrow j\ \})\ \big).$$

Thus, by Theorem 6.8 again, we have $a \in \mathrm{LC}(\varPhi)(x)$, indeed. **Q.e.d. (Theorem 10.3)**

Definition 10.4 (GC)
The *greatest-fixpoint operator* GC maps any class operator $\varPhi$ to a class operator $\mathrm{GC}(\varPhi)$ given by $\mathrm{GC}(\varPhi)(X) := \bigcup_{Z \subseteq \varPhi(Z)\setminus X} Z$, where $\mathrm{DGC}(\varPhi)(X) := \mathcal{V} \setminus \mathrm{GC}(\varPhi)(X)$, for $X \subseteq \mathcal{V}$.

Lemma 10.5 *If $\varPhi$ is a monotonic class operator, then* $\mathrm{GC}(\varPhi)$ *satisfies*
$$\mathrm{GC}(\varPhi)(X) \;=\; \varPhi(\mathrm{GC}(\varPhi)(X)) \setminus X$$
for all $X \subseteq \mathcal{V}$, and $\mathrm{DGC}(\varPhi)$ *is a closure operator.*

Proof of Lemma 10.5
Let $\varPhi$ be a monotonic class operator. Define $\varPhi_\setminus$ by $\varPhi_\setminus(X)(Y) := \varPhi(Y)\setminus X$. Then $\varPhi_\setminus(X)$ is a monotonic class operator as well, for every $X \subseteq \mathcal{V}$. Then, by Theorem 5.3 applied to $\varPhi_\setminus(X)$, we get that $\mathrm{GC}(\varPhi)(X)$ is the greatest fixpoint of $\varPhi_\setminus(X)$. Thus, $\mathrm{GC}(\varPhi)$ satisfies the equation stated in the lemma. Thus, $\mathrm{DGC}(\varPhi)$ is extensive. Moreover, $\mathrm{DGC}(\varPhi)$ is obviously a monotonic class operator. It remains to show $\mathrm{DGC}(\varPhi)\big(\ \mathrm{DGC}(\varPhi)(X)\ \big) \subseteq \mathrm{DGC}(\varPhi)(X)$ for every $X \subseteq \mathcal{V}$. This means that we have to show $\mathrm{GC}(\varPhi)\big(\ \mathrm{DGC}(\varPhi)(X)\ \big) \supseteq \mathrm{GC}(\varPhi)(X)$. This means that we have to show $\mathrm{GC}(\varPhi)\big(\ V\ \big) \supseteq W$ for $V := \mathrm{DGC}(\varPhi)(X)$ and $W := \mathrm{GC}(\varPhi)(X)$. This means to show $\bigcup_{Z \subseteq \varPhi(Z)\setminus V} Z \supseteq W$. For this it again suffices to show $W \subseteq \varPhi(W) \setminus V$. By definition of V and W, this means $W \subseteq \varPhi(W) \cap W$, i.e. $W \subseteq \varPhi(W)$, i.e. $\mathrm{GC}(\varPhi)(X) \subseteq \varPhi(\mathrm{GC}(\varPhi)(X))$, which has already been shown. **Q.e.d. (Lemma 10.5)**

There is no analog to Theorem 10.3 for $\mathrm{DGC}(\varPhi)$:

Example 10.6 (Algebraic $\varPhi$ with $\mathrm{DGC}(\varPhi)$ not even set-continuous)
For the algebraic closure operator $\varPhi$ of Example 6.11 the following holds: For all $x \in \mathfrak{P}(\mathcal{V})$: $\mathrm{GC}(\varPhi)(x) = \mathcal{O}\setminus x$, i.e. $\mathrm{DGC}(\varPhi)(x) = x \cup \mathcal{V}\setminus\mathcal{O}$. But: $\mathrm{GC}(\varPhi)(\{\ \beta \in \mathcal{O} \mid 2 \prec \beta\ \}) = \emptyset$, i.e. $\mathrm{DGC}(\varPhi)(\{\ \beta \in \mathcal{O} \mid 2 \prec \beta\ \}) = \mathcal{V}$. This means that $\mathrm{DGC}(\varPhi)$ is not set-continuous because of $\{0,1,2\} \subseteq \mathrm{DGC}(\varPhi)(\{\ \beta \in \mathcal{O} \mid 2 \prec \beta\ \}) \setminus \bigcup_{x \in \mathfrak{P}(\{\ \beta \in \mathcal{O}\ \mid\ 2 \prec \beta\ \})} \mathrm{DGC}(\varPhi)(x)$.

Remark 10.7 It is not too surprising that the greatest fixpoint of an algebraic class operator does not behave nicer than that of a set-continuous one (cf. Theorem 6.9 and Example 6.11) as is the case for the least fixpoint (cf. Theorem 6.8), because we probably need something co-algebraic for the greatest fixpoint. That the greatest-fixpoint operator of a set-continuous class operator is not set-continuous (cf. Example 10.6) contrary to the least-fixpoint closure operator (cf. Theorem 10.3) may just mean that — contrary to what was written in Remark 8.2 — we nevertheless need the dual concept of a closure operator. This should be investigated in the future. What about you, Peter?

Notes

Note 1 In [Forster, 2008] we read:

> "Quite early on QUINE added classes to NF to obtain a theory of sets-with-classes known as ML. In the ZF context, adding classes is a natural thing to do, for it enables one to reduce the infinite replacement scheme to a single set-existence axiom: "the image of a set in a class is a set" if augmented with suitable class-existence axioms. However, none of the axioms of NF refer to classes in the way the replacement scheme of ZF does, so there is nothing for the class existence axioms to do. For this reason ML is nowadays regarded as a pointless syntactic complication of NF with no new mathematics and is not the subject of any research."

This argument does not count, however, regarding the *application* of class theories, where ML is clearly preferable to NF, just as MK is clearly preferable to NBG and ZF, simply because you want to have an object for the extension of any predicate, even if you later find out that that object must be a proper class. This preference of class theories over mere set theories is especially strong for this paper, because we want to work with the common sub-theory of both ML and MK in general, choose one the two only if necessary, and, moreover, discuss class operators.

Note 2 On a first look, it may seem that there is a chance not take $\mathcal{U}(\_)$ as an elementary predicate symbol, but to *define* it via the class constructor, say as

$$\mathcal{U} := \{\ x \mid \neg\exists Z.\ (Z \in x) \quad \wedge \quad x \neq \emptyset\ \}.$$

This definition has two weaknesses:

1. "$\neg\exists Z.\ (Z \in x)$" may actually be a slight overspecification.
2. To restrict all urelements to be elements of $\mathcal{V}$ may actually be a slight overspecification.
3. This would force all urelements to receive the same number during stratification, which may actually be a slight overspecification.
4. More seriously, in § 1.6, we present a procedure to eliminate the class constructor from any given formula, where the elimination of exactly one class constructor as an argument of an $=$-atom has to introduce the symbol "$\mathcal{U}$" in an atomic formula of the form "$X \in \mathcal{U}$". If we have to replace "$\mathcal{U}$" with the definition suggested above, then this introduces two new class constructors. The outer one is no problem for the elimination procedure because it occurs to the right of the symbol "$\in$". The inner one results from the elimination of the defined symbol "$\emptyset$" and introduces an $=$-atom exactly of the type that we wanted to eliminate. Thus, the elimination procedure would loop and not terminate anymore. This means that we have to accept at least one of "$\mathcal{U}$", "$\emptyset$", and "$\{\ x \mid A\ \}$" as elementary. And our preference is definitely to take "$\mathcal{U}$" as elementary.

Note 3 "Ur" (speak: "oor" with "oo" as in "boot") is a German prefix which means "above the current construction" ("hinauf", "hinaus"), often also with the temporal aspect of "being generated in advance" (Latin: *primigenius*). Moreover this prefix means "original" ("ursprünglich"), and "not derived" ("unabgeleitet"). For instance, "Ur" turns a grandmother (Großmutter) into a great-grandmother (Urgroßmutter), and an ancestor (Ahn) into an ancestor whose ancestors are unknown (Urahn). For more information on the semantics of "Ur", cf. [Grimm & Grimm, 1854ff., Vol. 24, p. 2356ff.].

Note 4 As the lemma does not require $\leq$ to be a reflexive ordering, a formal proof also stops us from typical human errors. Thus, let us do it in the formal calculus of [Wirth, 2004]:

Expanding the definitions we get

$$\forall u.\ \big(\ \forall y.\ (\forall z.\ (P(z) \Rightarrow z{\geq}y)\ \Rightarrow\ y{\leq}u)\ \ \Rightarrow\ \ u{\geq}s\ \big)\quad\Rightarrow\quad \forall x.\ \big(\ P(x) \Rightarrow x{\geq}s\ \big),$$

and then

$$\forall u.\ \big(\ \forall y.\ (\forall z.\ (P(z) \Rightarrow y{\leq}z)\ \Rightarrow\ y{\leq}u)\ \ \Rightarrow\ \ s{\leq}u\ \big)\quad\Rightarrow\quad \forall x.\ \big(\ P(x) \Rightarrow s{\leq}x\ \big),$$

which reduces in an α-, a δ^{-}-, and another α-step to the sequent

$$\neg\forall u.\ \big(\ \forall y.\ (\forall z.\ (P(z) \Rightarrow y{\leq}z)\ \Rightarrow\ y{\leq}u)\ \ \Rightarrow\ \ s{\leq}u\ \big),\quad \neg P(x^{\delta^-}),\quad s{\leq}x^{\delta^-}.$$

Restricting to a multiplicity of 1, a γ-step (setting u to x^{δ^-}) reduces this to

$$\neg\big(\ \forall y.\ (\forall z.\ (P(z) \Rightarrow y{\leq}z)\ \Rightarrow\ y{\leq}x^{\delta^-})\ \ \Rightarrow\ \ s{\leq}x^{\delta^-}\ \big),\quad \neg P(x^{\delta^-}),\quad s{\leq}x^{\delta^-}.$$

A β-step reduces this to the two sequents

$$\forall y.\ (\forall z.\ (P(z) \Rightarrow y{\leq}z)\ \Rightarrow\ y{\leq}x^{\delta^-}),\quad \neg P(x^{\delta^-}),\quad s{\leq}x^{\delta^-}$$

and

$$s{\not\leq}x^{\delta^-},\quad \neg P(x^{\delta^-}),\quad s{\leq}x^{\delta^-}.$$

The second sequent is a tautology, and a δ^{-}- and an α-step reduce the first sequent to

$$\neg\forall z.\ (P(z) \Rightarrow y^{\delta^-}{\leq}z),\quad y^{\delta^-}{\leq}x^{\delta^-},\quad \neg P(x^{\delta^-}),\quad s{\leq}x^{\delta^-}$$

and a γ- and a β-step reduce this to the two tautologies

$$P(x^{\delta^-}),\quad y^{\delta^-}{\leq}x^{\delta^-},\quad \neg P(x^{\delta^-}),\quad s{\leq}x^{\delta^-}$$

$$y^{\delta^-}{\not\leq}x^{\delta^-},\quad y^{\delta^-}{\leq}x^{\delta^-},\quad \neg P(x^{\delta^-}),\quad s{\leq}x^{\delta^-}.$$

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