The Descriptive Operators iota, tau, and epsilon

— On their Origin, Partial Axiomatization, Model-Theoretic Semantics, Practical Applicability

Claus-Peter Wirth

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- Introduction:

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Descriptive Terms instead of Quantifiers

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  - inefficiency of computation (no incrementality, no in situ handling), e.g. in deep analysis of NL semantics.
- Mathematics often needs (quantified) free symbols, e.g. for Math. Induction (*Descente infinie*) or for computat. of solutions.

Quantifiers are Frege's artificial entities [1879] (not: quantification).

Essential Axiomatizations of ι, τ, ε

- $\exists! x^{\mathbb{B}}.\, A \;\Rightarrow\; A\{x^{\mathbb{B}} \mapsto \iota x^{\mathbb{B}}.\, A\}$ $\qquad (\iota_0)$

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  Consequence in Classical Logic $(\{A \mapsto \neg B\})$:

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- $\forall x^{\mathbb{B}}.\, B \;\Leftarrow\; B\{x^{\mathbb{B}} \mapsto \tau x^{\mathbb{B}}.\, B\}$ $\qquad (\tau_0)$

PART II

- Origins of ι, τ, ε
- How to Specify them
- Choose the ε
- History of Disappointed Expectations

On the History of Peano's ι

- Frege [1893] writes a boldface backslash.

  $\backslash\xi = x^{\mathbb{B}}$ if there is some $x^{\mathbb{B}}$ s.th. $\forall y^{\mathbb{B}}.\,(\xi(y^{\mathbb{B}}) \Leftrightarrow (x^{\mathbb{B}} = y^{\mathbb{B}}))$;
  $\backslash\xi = \xi$ otherwise.

- Peano writes "$\bar{\iota}$" [1896f.] or an inverted "ι" [1899, German].

  His "ι" is "$\iota(x) := \{x\}$", his "$\bar{\iota}$" is the inverse function of his "ι".

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- Hilbert & Bernays [1934] require the completion of a proof of $\exists! x^{\mathbb{B}}.\,A$ before the term $\iota x^{\mathbb{B}}.\,A$ may be formed.
- Quine and many others have ιs with explicit definitions.

Only Peano has always denoting terms + the intended partial spec.

Implicit Partial vs. Explicit Definition

- Peano with his preference on written languages for specification and communication (over calculi) stays within the proper limits:

  Avoid any Overspecification!

  (with all its unintended consequences)

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- Frege, Quine, &c. in the tradition of unconditional explicit definition (*“definiendum* $:=$ *definiens”*) (syntactic, always total):

  Eliminability!

  - Eliminability of the ι already requires absurdly powerful logical framework.
  - Eliminability impossible in principle for the ε in general, because of its indefiniteness.

On the History of Hilbert's τ

[Hilbert, 1923]: *Die logischen Grundlagen der Mathematik.*
Talk of Sept. 1922.

- τ stands for *transfinite function*
- $\tau : (i \rightarrow o) \rightarrow i, \qquad A : i \rightarrow o$.
- Transfinite Axiom: 11. $A(\tau A) \Rightarrow A(a)$.
- Acknowledgment for Paul Bernays in footnote:
  "Die Erkenntnis, daß die *eine* Formel 11. zur Herleitung dieser sämtlichen Formeln genügt, verdanke ich P. Bernays."
- Warning 1: A different function is the "τ" in Kneser's private notes to Hilbert's 1921/22 lecture *Grundlagen der Mathematik.*
- Warning 2: An "ε" is written for the τ in Kneser's private notes to Hilbert's 1922/23 lecture *Logische Grundlagen der Math.*
- Warning 3: Nicolas Bourbaki writes "τ", but it is an ε!

On the Early History of Hilbert's ε

- $A(a) \Rightarrow A(\varepsilon A)$ (deduct. equivalent to (ε_0)) is called
  - *transfinite axiom 1* (as binder) [Ackermann, 1925]
  - *axiom of choice* [Hilbert, 1926]
  - *logical ε-axiom* [Hilbert, 1928]
  - *ε-formula* (but as binder) [Hilbert & Bernays, 1939]
- ε is called the
  - *transfinite logical choice function* [Hilbert, 1926]
  - *logical ε-function* [Hilbert, 1928]
  - *ε-symbol* [Hilbert & Bernays, 1939]

[Ackermann, 1925]: *Begründung des „t.n.d." mittels der Hilbert'schen Theorie der Widerspruchsfreiheit.* Abstract PhD thesis 1924.

[Hilbert, 1926]: *Über das Unendliche.* Talk of June 1925.

[Hilbert, 1928]: *Die Grundlagen der Mathematik.* Talk of July 1927.

[Hilbert & Bernays, 1939]: *Grundlagen der Mathematik, Vol. II.*

The ε is The Choice in Practice

- The ι is no use unless $\exists! x^{\mathbb{B}}.\,A$.

  But given $\exists! x^{\mathbb{B}}.\,A$, (ι_0), (ε_0), we have:

  $$\iota x^{\mathbb{B}}.\,A \;=\; \varepsilon x^{\mathbb{B}}.\,A.$$

  Thus — to obtain weaker proof obligations — always use the ε instead of ι (unless eliminability relevant):

  - Less proof work!
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- τ instead of ε makes sense only in non-classical logics.
  No essential difference in classical logic.

Sharpened 1st ε-Theorem [H. & Bernays, 1939]

Given: a derivation of $\exists x_1^{\mathbb{B}}.\ \ldots \exists x_r^{\mathbb{B}}.\ A$

(containing no bound variables besides the ones bound by the prefix $\exists x_1^{\mathbb{B}}.\ \ldots \exists x_r^{\mathbb{B}}.$)

from the formulas $P_1, \ldots, P_k$

(containing neither formula variables nor bound variables)

in the predicate calculus

(incl. ε-formula and $=$-substitutability as axiom schemes, plus $=$-reflexivity).

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We can construct a (finite) disjunction of the form

$$\bigvee_{i=0}^{s}\ A\{x_1^{\mathbb{B}}, \ldots, x_r^{\mathbb{B}} \mapsto t_{i,1}, \ldots, t_{i,r}\}$$

and a derivation of it

- in which bound variables do not occur at all
- from $P_1, \ldots, P_k$ and $=$-axioms
  (containing neither formula variables nor bound variables)
- in the quantifier-free predicate calculus (i.e. tautologies plus
  the inference schema [of modus ponens] and the substitution rule).

Note that r, s range over natural numbers including 0, and that A, $t_{i,j}$, and P_i are ε-free because otherwise they would have to include (additional) bound variables.

ε — A History of Failures? (“twilight of his career”)

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- New *goal-directed* calculi [Herbrand, 1930] (*Modus Ponens*-free) and [Gentzen, 1935] (Cut-free) do not have an ε.
- Consistency proofs in [Herbrand, 1932] and in [Gentzen, 1936, 1938, 1943] do not use the ε.
- The displaced Ackermann finally [1940] proves the termination of an improved ε-substitution method in arithmetic, only to draw level w.r.t. consistency proofs with [Gentzen, 1938].

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- Kreisel’s [1958] *unwinding program* (Constructive equiv. math.?) is not properly formulated and followed.
- Leisenring’s textbook [1969] renders the ε as too impractical for computer sci. and autom. theorem proving.
- Several projects to translate ε’s main reference, Hilbert–Bernays *Grundlagen der Mathematik* [1934/39] [1968/70], failed.

PART III

- Via Overspecification toward a

  Model-Theoretic Semantics for the ε

- Choice: Have really indefinite choice
  + committed choice as an option

- Take Leisenring's Satisfiability,
  but not his Notion of Validity

Overspecification: Extensionality (E2)

Ackermann's (II,4), Bourbaki's (S7), Leisenring's (E2):

$$\forall x^{\mathbb{B}}.\ (A_0 \Leftrightarrow A_1) \qquad \Rightarrow \qquad \varepsilon x^{\mathbb{B}}.\ A_0 \ = \ \varepsilon x^{\mathbb{B}}.\ A_1$$

- Good: Syntactical differences between A_1 and A_2 should not matter. Deterrent example: [Asser, 1957, type-3]

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- Bad1: All classical theorems become intuitionistic ones.
- Bad2: Committed choice should be an option, not a must. "A bishop met a bishop."
- Actually committed choice *must* be an option:

$$\exists x^{\mathbb{B}}.\ (x^{\mathbb{B}} \neq x^{\mathbb{B}})$$

$$\varepsilon x^{\mathbb{B}}.\ (x^{\mathbb{B}} \neq x^{\mathbb{B}}) \neq \varepsilon x^{\mathbb{B}}.\ (x^{\mathbb{B}} \neq x^{\mathbb{B}})$$

$$0 \neq \varepsilon x^{\mathbb{B}}.\ (x^{\mathbb{B}} \neq x^{\mathbb{B}})$$

$$0 \neq 1$$

Model-Theoretic Semantics

- Motivation:
  Practical Applicability requires a Model-Theoretic Semantics.

- To define Satisfiability, the Existence of a (generalized) choice function
  works fine in the evaluation of ε-terms.

- ε-formula plus (E2) turn predicate calculus sound and complete for validity w.r.t. all possible choice functions,
  cf. [Asser, habil, 1957, type-1], [Leisenring, textbook, 1969]

Main Theses

Practical usefulness as well as formal adequacy of a straightforward model-theoretic specification of the ε require:

- Both satisfiability and validity must refer to the existence of choice functions, not to all of them:

$$\mathsf{Bishop}(\mathsf{Tebarz}) \models \ \varepsilon x^{\mathbb{B}}.\ \mathsf{Bishop}(x^{\mathbb{B}}) \ = \ \mathsf{Tebarz}$$

- Non-commitment to a choice must be possible where it is not required: $\models \ \varepsilon_i x^{\mathbb{B}}.\ \mathsf{Bishop}(x^{\mathbb{B}}) \ \neq \ \varepsilon_j x^{\mathbb{B}}.\ \mathsf{Bishop}(x^{\mathbb{B}})$

  (Indexed ε à la Heusinger/Egli)

Better without choice functions: $\models \ x_0^{\mathbb{V}} = \mathsf{Tebarz}, \qquad x_1^{\mathbb{V}} \neq x_2^{\mathbb{V}}$

with choice condition $C(x_i^{\mathbb{V}}) := \varepsilon x^{\mathbb{B}}.\ \mathsf{Bishop}(x^{\mathbb{B}})$ for $i \in \{0, 1, 2\}$.

ε-substitution becomes subst. of free variables $x_i^{\mathbb{V}}$.

PART IV: Wirth's Free-Variable Framework

- Quantification without Quantifiers
- Hilbert's ε
- Liberalized δ-rules
- Fermat's *Descente infinie*

Free Variables and Atoms

- Occur frequently in math & computer science
- Their function depends on context:
  varying, implicit, ad hoc
- Here:
  disjoint sets of symbols for different functions
- New:
  only two functions of free variables / atoms left:
  existentially / universally quantified.
- New:
  Henkin quantification can be modeled directly
  with positive/*negative* variable-conditions.

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- Semantics is uniquely expressed in (2) and (3).

Free Variables (=“meta”=“dummy”=”γ”=”δ^+”)

(1)

$$\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11 \\ 13 \end{pmatrix}$$

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- Semantics is uniquely expressed in (2) and (3).

Free Atoms

- $\mathbb{A}$, Universally quantified (implicitly)
- Arbitrary object in a discourse
- Atomic: black-box, no information on it ever
- Except: Is it an atom?
  Different from another atom?
- Origin of name:
  Set theories with atoms (or urelements)
- Instantiated locally and repeatedly in application of lemmas or induction hypotheses

Free Variables

- $\mathbb{V}$, Existentially quantified (implicitly)
- Place-holder in a discourse
- Gather and store information
- Replaced with a definition or a description
- Origin of name:
  Fitting's free-variable semantic tableaus
- Rigid: Instantiated globally, once and for all,
  with possible effect on input theorem

(Reductive) Inference (Smullyan's classification)

γ-rule:
$$\frac{\Gamma,\quad \exists x^{\mathbb{B}}.A,\quad \Delta}{A\{x^{\mathbb{B}}\mapsto x^{\mathbb{V}}\},\quad \Gamma,\quad \exists x^{\mathbb{B}}.A,\quad \Delta}$$

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δ^{-}-rule:

$$\frac{\Gamma,\ \forall y^{\mathbb{B}}.A,\ \Delta}{A\{y^{\mathbb{B}}\mapsto y^{\mathbb{A}}\},\ \Gamma,\ \Delta} \qquad \mathbb{V}(\Gamma, \forall y^{\mathbb{B}}.A, \Delta) \times \{y^{\mathbb{A}}\}$$

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$$\frac{\varGamma,\quad \exists x^{\mathbb{B}}.A,\quad \varDelta}{A\{x^{\mathbb{B}}{\mapsto}x^{\mathbb{V}}\},\quad \varGamma,\quad \exists x^{\mathbb{B}}.A,\quad \varDelta}$$

δ^--rule:

$$\frac{\varGamma,\ \forall y^{\mathbb{B}}.A,\ \varDelta}{A\{y^{\mathbb{B}}{\mapsto}y^{\mathbb{A}}\},\ \varGamma,\ \varDelta}$$

$$\mathbb{V}(\varGamma, \forall y^{\mathbb{B}}.A, \varDelta) \times \{y^{\mathbb{A}}\}$$

δ^+-rule:

$$\frac{\varGamma,\ \forall y^{\mathbb{B}}.A,\ \varDelta}{A\{y^{\mathbb{B}}{\mapsto}y^{\mathbb{V}}\},\ \varGamma,\ \varDelta}$$

$$(y^{\mathbb{V}},\quad \varepsilon y^{\mathbb{B}}.\ \neg A)$$

$$\mathbb{V}\mathbb{A}(\forall y^{\mathbb{B}}.A) \times \{y^{\mathbb{V}}\}$$

1st Example Proof Attempt

δ^--rule:

$$\frac{\Gamma,\ \forall y^{\mathbb{B}}.A,\ \ \Delta}{A\{y^{\mathbb{B}}\mapsto y^{\mathbb{A}}\},\ \ \Gamma,\ \ \Delta} \qquad \mathbb{V}(\Gamma, \forall y^{\mathbb{B}}.A, \Delta) \times \{y^{\mathbb{A}}\}$$

Proof task: $\exists x^{\mathbb{B}}.\forall y^{\mathbb{B}}.(x^{\mathbb{B}} = y^{\mathbb{B}})$

1st Example Proof Attempt

δ^--rule:

$$\frac{\Gamma,\ \forall y^{\mathbb{B}}.A,\ \ \Delta}{A\{y^{\mathbb{B}}\mapsto y^{\mathbb{A}}\},\ \ \Gamma,\ \ \Delta} \qquad \mathbb{V}(\Gamma,\forall y^{\mathbb{B}}.A,\Delta)\times\{y^{\mathbb{A}}\}$$

Proof task: $\exists x^{\mathbb{B}}.\forall y^{\mathbb{B}}.(x^{\mathbb{B}}=y^{\mathbb{B}})$

γ-step: $\forall y^{\mathbb{B}}.(x^{\mathbb{V}}=y^{\mathbb{B}}),\quad \exists x^{\mathbb{B}}.\forall y^{\mathbb{B}}.(x^{\mathbb{B}}=y^{\mathbb{B}})$

1st Example Proof Attempt

δ^--rule:

$$\frac{\Gamma,\ \forall y^{\mathbb{B}}.A,\ \Delta}{A\{y^{\mathbb{B}}\mapsto y^{\mathbb{A}}\},\ \Gamma,\ \Delta} \qquad \mathbb{V}(\Gamma,\forall y^{\mathbb{B}}.A,\Delta)\times\{y^{\mathbb{A}}\}$$

Proof task: $\exists x^{\mathbb{B}}.\forall y^{\mathbb{B}}.(x^{\mathbb{B}}=y^{\mathbb{B}})$

γ-step: $\forall y^{\mathbb{B}}.(x^{\mathbb{V}}=y^{\mathbb{B}}),\quad \exists x^{\mathbb{B}}.\forall y^{\mathbb{B}}.(x^{\mathbb{B}}=y^{\mathbb{B}})$

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$$\frac{\Gamma,\ \forall y^{\mathbb{B}}.A,\ \Delta}{A\{y^{\mathbb{B}}\mapsto y^{\mathbb{A}}\},\ \Gamma,\ \Delta} \qquad \mathbb{V}(\Gamma,\forall y^{\mathbb{B}}.A,\Delta)\times\{y^{\mathbb{A}}\}$$

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δ^--step: $(x^{\mathbb{V}}=y^{\mathbb{A}}),\quad \exists x^{\mathbb{B}}.\forall y^{\mathbb{B}}.(x^{\mathbb{B}}=y^{\mathbb{B}})$

Apply $\{x^{\mathbb{V}}\mapsto y^{\mathbb{A}}\}$?

1st Example Proof Attempt

δ^--rule:

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Proof task: $\exists x^{\mathbb{B}}.\forall y^{\mathbb{B}}.(x^{\mathbb{B}} = y^{\mathbb{B}})$

γ-step: $\forall y^{\mathbb{B}}.(x^{\mathbb{V}} = y^{\mathbb{B}}),\quad \exists x^{\mathbb{B}}.\forall y^{\mathbb{B}}.(x^{\mathbb{B}} = y^{\mathbb{B}})$

δ^--step: $(x^{\mathbb{V}} = y^{\mathbb{A}}),\quad \exists x^{\mathbb{B}}.\forall y^{\mathbb{B}}.(x^{\mathbb{B}} = y^{\mathbb{B}})$

Apply $\{x^{\mathbb{V}}{\mapsto}y^{\mathbb{A}}\}$?

Record dependency in
negative variable-condition N: $\quad (x^{\mathbb{V}}, y^{\mathbb{A}}) \in N$

2nd Example Proof Attempt

δ^+-rule:

$$\frac{\Gamma,\ \forall y^{\mathbb{B}}.A,\ \Delta}{A\{y^{\mathbb{B}}\mapsto y^{\mathbb{V}}\},\ \Gamma,\ \Delta} \qquad \begin{array}{l}(y^{\mathbb{V}},\ \ \varepsilon y^{\mathbb{B}}.\ \neg A)\\ \mathbb{VA}(\forall y^{\mathbb{B}}.A)\times\{y^{\mathbb{V}}\}\end{array}$$

Proof task: $\exists x^{\mathbb{B}}.\forall y^{\mathbb{B}}.(x^{\mathbb{B}}=y^{\mathbb{B}})$

γ-step: $\forall y^{\mathbb{B}}.(x^{\mathbb{V}}=y^{\mathbb{B}}),\ \ \exists x^{\mathbb{B}}.\forall y^{\mathbb{B}}.(x^{\mathbb{B}}=y^{\mathbb{B}})$

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Apply $\{x^{\mathbb{V}}\mapsto y^{\mathbb{V}}\}$?

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Apply $\{x^{\mathbb{V}}\mapsto y^{\mathbb{V}}\}$?

Record dependency in
positive variable-condition P: $(x^{\mathbb{V}},y^{\mathbb{V}})\in P$

2^{nd} Example Proof Attempt

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Apply $\{y^{\mathbb{V}}\mapsto x^{\mathbb{V}}\}$?

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Apply $\{y^{\mathbb{V}}\mapsto x^{\mathbb{V}}\}$?

We have to prove in advance $\{y^{\mathbb{V}}\mapsto x^{\mathbb{V}}\}$-instance of:

(ε_0) $\exists y^{\mathbb{B}}.\neg A \Rightarrow \neg A\{y^{\mathbb{B}}\mapsto y^{\mathbb{V}}\}$

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We have to prove in advance $\{y^{\mathbb{V}}\mapsto x^{\mathbb{V}}\}$-instance of:

(ε_0) $\quad \exists y^{\mathbb{B}}.\neg A \Rightarrow \neg A\{y^{\mathbb{B}}\mapsto y^{\mathbb{V}}\}$

i.e. $\quad \exists y^{\mathbb{B}}.\neg(x^{\mathbb{V}}=y^{\mathbb{B}}) \Rightarrow \neg(x^{\mathbb{V}}=x^{\mathbb{V}})$

Variable-Conditions in the Literature

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- Lincoln A. Wallen 1990:
  - Single positive relation
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Variable-Conditions in the Literature (contd.)

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    Unsound!
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    Automated Deduction, Vol. I, 1998]:
    Unsound!
- Wirth's previous versions with single relation
  - Negative relation [FTP'1998]
  - Positive relation with Variable reuse [J. IGPL 2004]
  - Standard positive relation [J. IGPL 2004], [J. Appl. L. 2008],
    SEKI-Report SR–2006–02 [2012]

Positive/Negative Variable-Condition (P, N)

$$\begin{array}{cccccc} P & \subseteq & (\mathbb{V} \uplus \mathbb{A}) & \times & \mathbb{V} \\ N & \subseteq & \mathbb{V} & \times & \mathbb{A} \end{array}$$

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- *Consistency:*

  Each cycle in the directed graph of $P \cup N$ has more than one edge from N.

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- *Consistency:*

  Each cycle in the directed graph of $P \cup N$ has more than one edge from N.

- *Admissible substitution* σ*:* $(P \cup D, N)$ is consistent.

  $(x^{\mathbb{VA}}, y^{\mathbb{V}}) \in D$ iff

  $y^{\mathbb{V}} \in \mathrm{dom}(\sigma)$ and

  $x^{\mathbb{VA}}$ is a free variable or free atom in $\sigma(y^{\mathbb{V}})$

Whole Proof Search Framework

DATA STRUCTURES:

- A Forest of and/or proof-attempt trees.

  Root of each tree carries an [open] proposition.

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PS-INVARIANT (“Preservation of solutions”):
“The the solutions of the leaves solve the root.”

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The root $(C, (P, N))$-reduces to the leaves.

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"The the solutions of the leaves solve the root."

The root $(C, (P, N))$-reduces to the leaves.

This is more than soundness of problem reduction!

Whole Proof Search Framework (contd.)

OPERATIONS:

- New conjectures get trivial proof-attempt tree.

Whole Proof Search Framework (contd.)

OPERATIONS:

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Whole Proof Search Framework (contd.)

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  $x^{\mathbb{V}} \neq y^{\mathbb{V}}$ means that the universe is non-trivial.

Whole Proof Search Framework (contd.)

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- New conjectures get trivial proof-attempt tree.
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  either deductively or inductively.
- Global substitution of rigid variables.
  PS-INVARIANT! But:
  $x^{\mathbb{V}} \neq y^{\mathbb{V}}$ means that the universe is non-trivial.
  It becomes false after application of $\{x^{\mathbb{V}} \mapsto y^{\mathbb{V}}\}$

Henkin Quantification

- “Every woman could love someone and
  every man could love someone,
  such that these loved ones could love each other.”

Henkin Quantification

- "Every woman could love someone and every man could love someone, such that these loved ones could love each other."
- As a Henkin-quantified IF-logic formula:

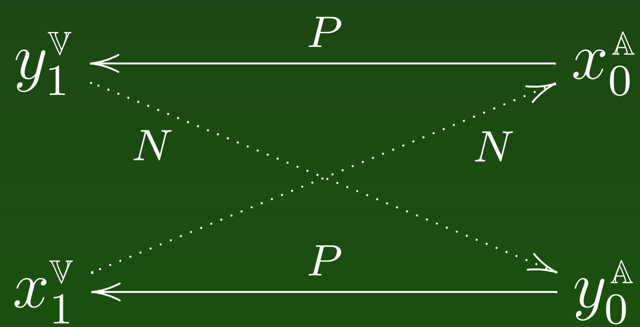
$$\forall x_0^{\mathbb{B}}.\ \forall y_0^{\mathbb{B}}.\ \left(\begin{array}{rl} & \left(\begin{array}{rl} & \mathsf{Female}(x_0^{\mathbb{B}}) \\ \wedge & \mathsf{Male}(y_0^{\mathbb{B}}) \end{array} \right) \\ \Rightarrow & \exists y_1^{\mathbb{B}}/y_0^{\mathbb{B}}.\ \exists x_1^{\mathbb{B}}/x_0^{\mathbb{B}}.\ \left(\begin{array}{rl} & \mathsf{Loves}(x_0^{\mathbb{B}}, y_1^{\mathbb{B}}) \\ \wedge & \mathsf{Loves}(y_0^{\mathbb{B}}, x_1^{\mathbb{B}}) \\ \wedge & \mathsf{Loves}(y_1^{\mathbb{B}}, x_1^{\mathbb{B}}) \\ \wedge & \mathsf{Loves}(x_1^{\mathbb{B}}, y_1^{\mathbb{B}}) \end{array} \right) \end{array} \right)$$

Henkin Quantification

- “Every woman could love someone and
every man could love someone,
such that these loved ones could love each other.”

- Represented in our Framework:

$$\left(\begin{array}{ll} & \mathsf{Female}(x_0^{\mathbb{A}}) \\ \wedge & \mathsf{Male}(y_0^{\mathbb{A}}) \end{array}\right) \Rightarrow \left(\begin{array}{ll} & \mathsf{Loves}(x_0^{\mathbb{A}}, y_1^{\mathbb{V}}) \\ \wedge & \mathsf{Loves}(y_0^{\mathbb{A}}, x_1^{\mathbb{V}}) \\ \wedge & \mathsf{Loves}(y_1^{\mathbb{V}}, x_1^{\mathbb{V}}) \\ \wedge & \mathsf{Loves}(x_1^{\mathbb{V}}, y_1^{\mathbb{V}}) \end{array}\right)$$



Binders are Bad:

- Quantifiers and the ε mess up formulas
- Quantifiers and the ε make reasoning difficult
- Quantifiers enforce a too primitive form of scoping
- The ε-binder produces terms of unmanageable size

Binding without Binders is Great:

- Free variables and atoms are what we need to manage practical applications
- Positive/Negative Variable-Conditions enable sophisticated scoping
- The term-sharing of free variables admits ε-binding that is manageable w.r.t. term size
- Our semantics for the ε is existential (!) and admits indefinite committed choice
- Free atoms admit mathematical induction in the liberal style of Fermat's *Descente Infinie*

Conclusion

Want your reasoning applications to be successful in practice?

- Ask for a tailored version of a free-variable framework!
- Do not introduce free variables and atoms ad hoc for operational purposes,
  but give them a clear semantics
- Get *both* free variables *and* free atoms:
- The liberalized δ-rule (δ^{+}-rule) is a practical improvement only if the non-liberalize δ-rule (δ^{-}-rule) remains available (Henkin, Fermat)
- Use an ε with existential semantics

Related Publications

- *Descente Infinie + Deduction*

  Logic J. of the IGPL 12:1–96, 2004, Oxford Univ. Press

- *Hilbert's epsilon as an Operator of Indefinite Committed Choice (IDC)*

  J. Applied Logic 6:287–317, 2008, Elsevier

- *A Simplified and Improved Free-Variable Frameword of Hilbert's ε as an Operator of IDC*

  SEKI-Report SR–2011–01, Revised May 2015.

  `http://arxiv.org/abs/1104.2444`

- *Hilbert–Bernays: Grundlagen der Mathematik*.

  1st English translation. With comments and German facsimile.

  `http://wirth.bplaced.net/p/hilbertbernays`

Semantic treatment of Variable-Conditions

$$\mathsf{e}(\pi)(\delta)(x^{\mathbb{V}}) := \pi(x^{\mathbb{V}})({}_{S_{\pi}\langle\{x^{\mathbb{V}}\}\rangle}\!\upharpoonleft\!\delta).$$

$$\pi : \mathbb{V} \rightsquigarrow (\mathbb{A} \rightsquigarrow \mathcal{S}) \rightsquigarrow \mathcal{S}, \qquad \delta : \mathbb{A} \rightsquigarrow \mathcal{S}, \qquad x \in \mathbb{V}$$

$$\mathsf{e} \quad : \quad (\mathbb{V} \rightsquigarrow (\mathbb{A} \rightsquigarrow \mathcal{S}) \rightsquigarrow \mathcal{S}) \quad \longrightarrow \quad (\mathbb{A} \rightsquigarrow \mathcal{S}) \quad \longrightarrow \quad \mathbb{V} \quad \rightsquigarrow \quad \mathcal{S}$$

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$$S_\pi \;:=\; \{\ (y^{\mathbb{A}}, x^{\mathbb{V}}) \mid x^{\mathbb{V}} \in \mathrm{dom}(\pi) \wedge y^{\mathbb{A}} \in \mathrm{dom}(\bigcup(\mathrm{dom}(\pi(x^{\mathbb{V}}))))\ \}$$

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π is $\mathcal{S}$-*compatible with* $(C, (P, N))$ if ... and

$(P \cup S_\pi, N)$ is consistent and

π respects C in $\mathcal{S}$

Reduction, Def.

G_0 $(C, (P, N))$*-reduces to* G_1 *in* $\mathcal{S}$ if

Reduction, Def.

Let G_0 and G_1 be sets of sequents.

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G_0 *$(C, (P, N))$-reduces to G_1 in $\mathcal{S}$* if

Reduction, Def.

Let G_0 and G_1 be sets of sequents.
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G_0 $(C, (P, N))$*-reduces to* G_1 *in* $\mathcal{S}$ if
for each π that is $\mathcal{S}$-compatible with $(C, (P, N))$:

if G_1 is $(\pi, \mathcal{S})$-valid,
then G_0 is $(\pi, \mathcal{S})$-valid.

Reduction PS-Invariant under Substitution

If G_0 $(C, (P, N))$-reduces to G_1 in $\mathcal{S}$,
then $G_0\sigma$ $(C', (P', N'))$-reduces to $G_1\sigma \cup (\langle O \rangle Q_C)\sigma$
in $\mathcal{S}$.

Reduction PS-Invariant under Substitution

For an (P, N)-substitution σ on $\mathbb{V}$,
for the extended σ-update $(C', (P', N'))$
of $(C, (P, N))$:

If G_0 $(C, (P, N))$-reduces to G_1 in $\mathcal{S}$,
then $G_0\sigma$ $(C', (P', N'))$-reduces to $G_1\sigma \cup (\langle O \rangle Q_C)\sigma$
in $\mathcal{S}$.

Example

In case of $C(y^{\mathbb{V}}) = \lambda v_0^{\mathbb{B}}.\ \varepsilon y^{\mathbb{B}}.\ (v_0^{\mathbb{B}} = y^{\mathbb{B}}{+}1)$:

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$Q_C(y^{\mathbb{V}})$

$$= \quad \forall v_0^{\mathbb{B}}. \left(\begin{array}{ll} & \exists y^{\mathbb{B}}.\ (v_0^{\mathbb{B}} = y^{\mathbb{B}}{+}1) \\ \Rightarrow & (v_0^{\mathbb{B}} = y^{\mathbb{V}}(v_0^{\mathbb{B}}){+}1) \end{array} \right)$$

Example

In case of $C(y^{\mathbb{V}}) = \lambda v_0^{\mathbb{B}}.\ \varepsilon y^{\mathbb{B}}.\ (v_0^{\mathbb{B}} = y^{\mathbb{B}}{+}1)$:

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$$= \quad \forall v_0^{\mathbb{B}}. \left(\begin{array}{ll} & \exists y^{\mathbb{B}}.\ (v_0^{\mathbb{B}} = y^{\mathbb{B}}{+}1) \\ \Rightarrow & (v_0^{\mathbb{B}} = y^{\mathbb{V}}(v_0^{\mathbb{B}}){+}1) \end{array} \right)$$

$$(Q_C(y^{\mathbb{V}}))\{y^{\mathbb{V}}{\mapsto}\mathsf{p}\}$$

$$= \quad \forall v_0^{\mathbb{B}}. \left(\begin{array}{ll} & \exists y^{\mathbb{B}}.\ (v_0^{\mathbb{B}} = y^{\mathbb{B}}{+}1) \\ \Rightarrow & (v_0^{\mathbb{B}} = \mathsf{p}(v_0^{\mathbb{B}}){+}1) \end{array} \right)$$